A Justification of Constant Mix Investment Strategies

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Abstract

The aim of this paper is to justify the use of constant mix investment strategies as an approximation for periodically rebalanced investment strategies. In Dhaene et al. (2005), general optimization problems are solved in a lognormal framework by deriving convex order bounds based on comonotonicity. The multi-period optimal portfolio selection problems discussed in Dhaene et al. (2005) are solved working in the class of constant mix strategies. As this class requires the investment portfolio to be rebalanced on a continuous basis, it can not be applied in practice. However, constant mix strategies have clear computational advantages compared to the more realistic class of periodically rebalanced strategies. Therefore it is important to investigate the appropriateness of these constant mix strategies, which is the main goal of this paper. Our numerical results confirm that, within our lognormal framework, continuous rebalancing can be seen as a limiting case of periodic rebalancing. Moreover, we see that results obtained by periodically rebalancing the available assets are very close to those obtained through continuously rebalancing them, which justifies the use of constant mix strategies.

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1 Introduction

In Dhaene et al. (2005), multi-period optimal portfolio selection problems in a Black & Scholes (1973) lognormal setting are extensively discussed. Based on results in Dhaene et al. (2002a), convex upper and lower bounds based on the concept of comonotonicity are derived to approximate sums of dependent lognormal random variables. The lower bound

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approximation in particular supplies very accurate and easy to compute approximations of the exact distribution. The results of Dhaene et al. (2005) are extended to more general optimal portfolio selection problems in two papers by Van Weert et al. (2010a & 2010b). Throughout the aforementioned papers, several optimization problems determining the optimal allocation of wealth are solved within the class of so-called *constant mix* strategies. In this class, an initial investment strategy is fixed at the beginning of the investment period, and, by continuously rebalancing the assets, the proportion of the total wealth invested in each asset class is kept constant throughout the entire investment horizon under consideration. In order to keep the fractions at their initial level, assets have to be bought and/or sold continuously, at each time instant.

As observed in the aforementioned papers, and as confirmed by the results in this paper, continuous rebalancing has great theoretical and computational advantages. In practice, however, continuous rebalancing is unfeasible, as it is impossible to buy and/or sell assets on a continuous basis. In general, practitioners can only periodically rebalance their portfolio, i.e. daily or even weekly or monthly. Large institutional investors such as pension funds might even prefer to rebalance their assets only once or twice per year. In this paper we examine how realistic it is to use continuous rebalancing as an approximation for periodic rebalancing. First of all we illustrate that continuous rebalancing can be seen as the limiting case of periodic rebalancing. Secondly we show with numerical examples that results obtained by continuously rebalancing the assets are very close to the periodic case, even when rebalancing is performed only once per year. Our results justify the use of constant mix strategies as considered e.g. in Dhaene et al. (2005). A confirmation of our results can be found in Kuhn & Luenberger (2010), where it is argued that continuous rebalancing only slightly outperforms discrete rebalancing if there are no transaction costs and if the rebalancing intervals are shorter than about one year.

Section 2 briefly describes the lognormal framework in which we will work. Section 3 explains the concepts of risk measures and comonotonicity. Next, Section 4 discusses the class of continuously rebalanced investment strategies, or constant mix strategies. Section 5 focuses on the discrete case, looking at investment strategies where periodic rebalancing is applied. To conclude, Section 6 gives numerical examples, illustrating the appropriateness of continuous rebalancing as an approximation for periodic rebalancing.

2 Lognormal framework

Throughout this paper we assume the classical continuous-time framework of Merton (1971), also known as the Black & Scholes (1973) setting. In our examples, we assume that we have $m \geq 2$ risky asset classes available in which we can invest, and that there is no risk-free asset class available. Denote the proportion invested in risky asset class i as π_i and the vector describing the portfolio as $\underline{\pi}^T = (\pi_1, \ldots, \pi_m)$. We assume that $\sum_{i=1}^{m} \pi_i = 1$. Although our results also hold in the general case, we assume short-selling is not allowed, which means $0 \leq \pi_i \leq 1$ for all i. See e.g. Björk (1998) for more details on the Black & Scholes setting. Throughout this paper we use the same notations and terminology as in Dhaene et al. (2005).

As both the time period and the investment horizon that we consider are typically long, the use of a Gaussian model for the stochastic returns may be justified by Central Limit Theorem arguments, see e.g. Cesari & Cremonini (2003) and Levy (2004) for some empirical evidence.

Throughout this paper, we use time units of one year, and an investment horizon of n years. The random variable Y_i^j denotes the return in year i of asset class j: an amount of 1 invested at time $i-1$ in asset class j will grow to $e^{Y_i^j}$ at time i. We assume yearly returns are normally distributed, with:

$$Y_i^j \sim N\left(\mu_j - \frac{\sigma_j^2}{2}, \sigma_j^2\right), \qquad i = 1, ..., n,\ j = 1, ..., m. \tag{1}$$

For a fixed asset class j, the random variables Y_i^j, $i \geq 1$, are assumed i.i.d. The return in a given year i is independent of the return in year $l \neq i$:

$$\mathrm{Cov}(Y_i^j, Y_l^k) = 0 \qquad i \neq l. \tag{2}$$

In a given year i, however, the returns of the different asset classes are correlated, with:

$$\mathrm{Cov}(Y_i^j, Y_i^k) = \sigma_{jk}, \qquad i = 1, ..., n. \tag{3}$$

We denote the drift vector and the variance-covariance matrix of the risky assets by $\underline{\mu}^T = (\mu_1, \ldots, \mu_m)$, and $\underline{\Sigma}$ respectively, with $\sigma_i^2 = \sigma_{ii}$. We assume that $\underline{\Sigma}$ is positive definite, which means that for all non-zero vectors $\underline{\pi}$ it holds that $\underline{\pi}^T \cdot \underline{\Sigma} \cdot \underline{\pi} > 0$. This assumption implies that all σ_i are strictly positive, and that $\underline{\Sigma}$ is non-singular.

In this paper we work in a *saving* and *terminal wealth* context: we look at the accumulated value of a series of invested amounts. All results, however, can readily be translated to a context of provisioning, where the quantity of interest is the present value of a series of payments or liabilities. Also, we assume yearly savings: each year a deterministic amount is saved to an account. Suppose we have given a series of deterministic, non-negative saving amounts $\alpha_0, \alpha_1, ..., \alpha_n$. Our aim is to evaluate the *terminal wealth*, or amount of money available on the account at time n. This random variable, which clearly depends on the investment strategy $\underline{\pi}$, will be denoted as $W_n(\underline{\pi})$.

Suppose we choose an initial investment strategy $\underline{\pi}$, according to which we will buy and sell assets during the whole investment period under consideration. The simplest class of strategies is the class of so-called *buy-and-hold* strategies, where all investments α_i are invested according to the initial strategy $\underline{\pi}$, and no rebalancing is performed during the investment period. Buy-and-hold strategies can be considered "do nothing" strategies: no matter what happens to relative values, no rebalancing is ever required. This class of strategies has been studied in a context of comonotonic approximations and convex order bounds in Marín-Solano et al. (2010).

As explained in the introduction, the focus in this paper is on investment strategies where rebalancing is performed, meaning that the available wealth is reinvested according to a predetermined investment strategy $\underline{\pi}$. In Section 4, continuous rebalancing is discussed, and a convex bound for the terminal wealth is derived. Section 5 gives a similar analysis for periodic rebalancing.

3 Risk measures and comonotonicity

In this section, to make our paper more self-contained, we briefly introduce the concepts of comonotonicity and risk measures, and their relationship. For more detailed information on the theory of comonotonicity and its many applications in actuarial science and finance we refer to Dhaene et al. (2002a,b) and Dhaene et al. (2008). More information on risk measures can be found e.g. in Kaas et al. (2008) or Denuit et al. (2005). A proof of Theorem 1 and more information about the relationship between risk measures and comonotonicity can be found in Dhaene et al. (2006).

A random vector $\underline{X} = (X_1, X_2, ..., X_n)$ is said to be *comonotonic* if:

$$\underline{X} \stackrel{d}{=} (g_1(Z), g_2(Z), ..., g_n(Z)) \tag{4}$$

for some common random variable Z and non-decreasing (or non-increasing) functions g_i. In other words, $\underline{X}$ is comonotonic if the individual variables X_i behave as non-decreasing functions (or all behave as non-increasing functions) of the same random variable. The components of the random vector (4) are maximally dependent, as increasing the outcome of Z leads to a simultaneous increase in the different outcomes of $g_i(Z)$.

Comonotonicity of $\underline{X}$ can also be characterized by

$$\underline{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_n}^{-1}(U)\right), \tag{5}$$

with U uniformly distributed on the unit interval. A finite sum of random variables is called a *comonotonic sum* if the terms in the sum form a comonotonic vector.

Throughout this paper we assume to be working with (conditioning) random variables such that all (conditional) expectations that are used are well-defined and finite.

Suppose we have a set of random variables representing the risks at hand. A *risk measure* is defined as a mapping from this set to the real numbers. A risk measure associated with a random variable X is commonly denoted as $\rho[X]$. The risk measure $\rho[X]$ summarizes the distribution function of the random variable X in a single real number. Any risk measure ρ should quantify the riskiness of X: the larger $\rho[X]$, the more 'dangerous' the risk X.

The most commonly used risk measure is the quantile risk measure, or Value-at-Risk (VaR). The VaR at level p, denoted by $Q_p(X)$ or $VaR_p(X)$, is defined as:

$$Q_p(X) = VaR_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in (0,1), \tag{6}$$

with $F_X(x) = \Pr(X \leq x)$ the cumulative distribution function of X. By convention, we take $\inf \varnothing = +\infty$.Value-at-Risk can be used to determine how much can be lost with a given probability over a predetermined time horizon, and measures the worst expected loss under normal market conditions over a specific time interval..

Other well-known risk measures are for example Tail Value-at-Risk (TVaR), Conditional Tail Expectation (CTE) and Expected Shortfall (ESF).

Combining risk measures and the comonotonic dependency structure, the following result is crucial in our setting:

Theorem 1 (Additivity of quantile risk measure for sums of comonotonic risks) *If the random vector $\underline{X} = (X_0, X_1, ..., X_n)$ is comonotonic, we have that*

$$Q_p\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Q_p\left(X_i\right), \tag{7}$$

for all $p \in (0,1)$.

This additivity property holds in general for all distortion risk measures, such as Value-at-Risk and Tail Value-at-Risk, see e.g. Denuit et al. (2005)

4 Continuous rebalancing

In Dhaene et al. (2005), the class of *constant mix* strategies, in which the fractions invested in the different asset classes are kept constant over time by continuously rebalancing the assets, has been analyzed in detail. To keep investments at this constant mix, the investor has to buy and/or sell assets continuously, at each time instant. These strategies are also referred to as constant proportional investment strategies. As stated in Dhaene et al. (2005), a constant mix strategy implies a 'buy low and sell high' rule in the sense that price and asset-purchase are counter-varying: if the price of a single asset goes up while the prices of the other assets remain constant, the proportion invested in the single asset should be decreased, and vice versa. Optimality of constant mix strategies in a utility theory setting is considered e.g. in Merton (1971).

Within our lognormal setting, where yearly returns are modelled by (1), (2) and (3), it can be shown that, if the portfolio is continuously rebalanced such that the investment proportions are kept constant, the portfolio return is also lognormally distributed. This was derived in Merton (1971, 1990), see also Rubinstein (1991), using stochastic arguments and Itô's Lemma. Milevsky & Posner (1998) derived the same result using more elementary arguments, by taking limits of lognormal sums. Furthermore, the result from Milevsky & Posner (1998) shows that, in a lognormal world, constant mix strategies can be seen as a limiting case of periodic rebalancing strategies, where rebalancing is performed at discrete points in time.

The drift vector and volatility corresponding to an investment portfolio $\underline{\pi}$ are written as $\mu(\underline{\pi})$ and $\sigma^2(\underline{\pi})$. In case of continuous rebalancing, it holds that

$$\mu(\underline{\pi}) = \underline{\pi}^T \cdot \underline{\mu} \text{ and } \sigma^2(\underline{\pi}) = \underline{\pi}^T \cdot \underline{\Sigma} \cdot \underline{\pi}. \tag{8}$$

The yearly returns $Y_i(\underline{\pi})$ of an investment portfolio $\underline{\pi}$ are independent and normally distributed random variables, with expected value $\mathrm{E}[Y_i(\underline{\pi})] = \mu(\underline{\pi}) - \frac{1}{2}\sigma^2(\underline{\pi})$ and variance $\mathrm{Var}[Y_i(\underline{\pi})] = \sigma^2(\underline{\pi})$.

When applying continuous rebalancing, the terminal wealth, denoted as $W_n^C(\underline{\pi})$, becomes:

$$W_n^C(\underline{\pi}) = \sum_{i=0}^{n} \alpha_i \; e^{\sum_{j=i+1}^{n} Y_j(\underline{\pi})}. \tag{9}$$

We will use a comonotonic lower bound to approximate $W_n^C(\underline{\pi})$, which we denote as $W_n^{C,l}(\underline{\pi})$. This approximation is a conditional expected value: $W_n^{C,l}(\underline{\pi}) = E\left[W_n^C(\underline{\pi}) \mid \Lambda\right]$. For any random variable Λ, $W_n^{C,l}(\underline{\pi})$ is a lower bound for $W_n^C(\underline{\pi})$ in the convex order sense:

$$W_n^C(\underline{\pi}) \geq_{cx} W_n^{C,l}(\underline{\pi}) = E\left[W_n^C(\underline{\pi}) \mid \Lambda\right]. \tag{10}$$

This means, by definition of convex order, that $E[W_n^C(\underline{\pi})] = E[W_n^{C,l}(\underline{\pi})]$ and that $W_n^{C,l}(\underline{\pi})$ has lower stop-loss premiums than $W_n^C(\underline{\pi})$. The conditioning random variable Λ is typically chosen as a linear combination of the yearly returns $Y_j(\underline{\pi})$. As explained in Dhaene et al. (2005), maximizing (an appropriate approximation of) the variance of $W_n^{C,l}(\underline{\pi})$ leads to the optimal Λ, and results in the following approximation:

$$W_n^{C,l}(\underline{\pi}) = \sum_{i=0}^{n} \alpha_i \; e^{(n-i)\ \mu(\underline{\pi}) - \frac{1}{2} r_i^2(\underline{\pi})\ (n-i)\ \sigma^2(\underline{\pi}) + r_i(\underline{\pi})\sqrt{n-i}\ \sigma(\underline{\pi})\ \Phi^{-1}(U)}, \tag{11}$$

with Φ the cdf of the standard normal distribution, U uniformly distributed on the unit interval and $r_i(\underline{\pi})$ the correlation between Λ and $\sum_{j=i+1}^{n} Y_j(\underline{\pi})$. The coefficients $r_i(\underline{\pi})$ are given by

$$r_i(\underline{\pi}) = \frac{\sum_{j=i+1}^{n} \sum_{k=0}^{j-1} \alpha_k \ e^{-k\ \mu(\underline{\pi})}}{\sqrt{n-i}\sqrt{\sum_{j=1}^{n} \left(\sum_{k=0}^{j-1} \alpha_k \ e^{-k\ \mu(\underline{\pi})}\right)^2}}. \tag{12}$$

Since the cash-flows α_k, $k = 0, ..., n$, are non-negative, the coefficients (12) are also non-negative. This means that all the terms in the sum (11) are non-decreasing functions of U, and hence (11) is a comonotonic sum. In this case we call $W_n^{C,l}(\underline{\pi})$ a *comonotonic lower bound*. The main advantage of this comonotonic dependency structure is the additivity property stated in Theorem 1, which makes it straightforward to determine the distribution function of our approximations $W_n^{C,l}(\underline{\pi})$. For example, the quantiles can be determined by simply replacing the uniform variable U by the probability level p:

$$Q_p\left[W_n^{C,l}(\underline{\pi})\right] = \sum_{i=0}^{n} \alpha_i \; e^{(n-i)\ \mu(\underline{\pi}) - \frac{1}{2} r_i^2(\underline{\pi})\ (n-i)\ \sigma^2(\underline{\pi}) + r_i(\underline{\pi})\sqrt{n-i}\ \sigma(\underline{\pi})\ \Phi^{-1}(p)}. \tag{13}$$

We know from e.g. Dhaene et al. (2002b) that the comonotonic lower bound as given by (11) is a very accurate approximation of $W_n^C(\underline{\pi})$.

In the following section, the class of periodic rebalancing investment strategies is discussed.

5 Periodic rebalancing

In this section, we focus on the class of investment strategies where *periodic rebalancing* is performed: at periodic points in time, e.g. each month or year, the investment portfolio is rebalanced. We divide each year into p periods. Hence, in total we have np periods at the end of which the investment portfolio is rebalanced, meaning that the available wealth is reinvested according to a predetermined investment strategy $\underline{\pi}$. As stated in the previous section, constant mix strategies arise as the limiting case of periodic rebalancing investment strategies by letting p go to infinity.

For given i, j and k, the random variable $Y_{i,k}^j$ models the return of asset class j in the k-th period of year i, or the return in the time interval $\left(i-1+\frac{k-1}{p}, i-1+\frac{k}{p}\right)$. In other words, an amount of 1 invested in asset class j at time $i-1+\frac{k-1}{p}$ will grow to $e^{Y_{i,k}^j}$ at time $i-1+\frac{k}{p}$. The random variables $Y_{i,k}^j$ are normally distributed, with

$$Y_{i,k}^j \sim N\left(\frac{1}{p}\left(\mu_j - \frac{\sigma_j^2}{2}\right), \frac{\sigma_j^2}{p}\right), \qquad i=1,...,n,\ j=1,...,m,\ k=1,...,p. \tag{14}$$

In the special case of yearly rebalancing, (14) reduces to (1).

Following the investment strategy $\underline{\pi}$, the return of the portfolio in year i is in this case given by the random variable:

$$S_i(\underline{\pi}) = \prod_{k=1}^{p}\left(\sum_{j=1}^{m}\pi_j e^{Y_{i,k}^j}\right). \tag{15}$$

An amount of 1 invested at time $i-1$, according to the investment strategy $\underline{\pi}$, will grow to $S_i(\underline{\pi})$ at time i. Note that $S_i(\underline{\pi})$ is a product of sums of correlated lognormal random variables.

As explained in the previous section, we assume we have given a series of yearly deterministic, non-negative saving amounts $\alpha_0, \alpha_1, ..., \alpha_n$. Using periodic rebalancing, the *terminal wealth*, denoted by $W_n^P(\underline{\pi})$, equals:

$$W_n^P(\underline{\pi}) = \sum_{i=0}^{n}\alpha_i \prod_{j=i+1}^{n} S_j(\underline{\pi}) = \left[\left(\left(\alpha_0 S_1(\underline{\pi}) + \alpha_1\right)S_2(\underline{\pi}) + \alpha_2\right)S_3(\underline{\pi}) + ...\right]S_n(\underline{\pi}) + \alpha_n. \tag{16}$$

Since, for each i, $S_i(\underline{\pi})$ is a product of sums of dependent random variables, it is impossible to determine the distribution function of $S_i(\underline{\pi})$ or $W_n^P(\underline{\pi})$ analytically. To solve this, we use convex order approximations based on the concept of comonotonicity.

To avoid further notational complexity, we assume in the remainder of this section that yearly rebalancing is used, or $p = 1$. Approximations in case $p > 1$ can be derived in a similar way.

We use a convex lower bound to approximate $S_i(\underline{\pi})$, which we denote as $S_i^l(\underline{\pi})$. As in the previous section, this approximation is a conditional expected value: $S_i^l(\underline{\pi}) = E\left[S_i(\underline{\pi}) \mid \Lambda_i\right]$. For each random variable Λ_i, $S_i^l(\underline{\pi})$ is a lower bound in the convex order sense:

$$S_i(\underline{\pi}) \geq_{cx} S_i^l(\underline{\pi}) = E\left[S_i(\underline{\pi}) \mid \Lambda_i\right]. \tag{17}$$

The conditioning random variable Λ_i is typically chosen as a linear combination of the yearly returns Y_i^j:

$$\Lambda_i = \sum_{j=1}^{m}\beta_j Y_i^j. \tag{18}$$

As explained in Dhaene et al. (2005), maximizing (an appropriate approximation of) the variance of $S_i^l(\underline{\pi})$ leads to the optimal Λ_i. Here this procedure results in coefficients β_j equal to $\pi_j e^{\mu_j}$. Since this optimal conditioning variable is dependent on the investment portfolio $\underline{\pi}$, we use the notation $\Lambda_i(\underline{\pi})$. We get

$$\Lambda_i(\underline{\pi}) = \sum_{j=1}^{m} \pi_j e^{\mu_j} Y_i^j. \tag{19}$$

Using these optimal conditioning random variables, we get the following lower bound approximations for $i = 1, ..., n$:

$$S_i^l(\underline{\pi}) = E\left[S_i(\underline{\pi}) \mid \Lambda_i(\underline{\pi})\right] = \sum_{j=1}^{m} \pi_j e^{\mu_j - \frac{1}{2} r_j^2 \sigma_j^2 + r_j(\underline{\pi})\sigma_j \Phi^{-1}(U_i)}, \tag{20}$$

with Φ the cdf of the standard normal distribution, U_i uniformly distributed on the unit interval, and $r_i(\underline{\pi})$ the correlation between $\Lambda_i(\underline{\pi})$ and Y_i^j. These coefficients can be shown to be equal to

$$r_i(\underline{\pi}) = \frac{\sum_{j=1}^{m} \pi_j e^{\mu_j} \sigma_{ij}}{\sigma_i \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} \pi_i \pi_j e^{\mu_i + \mu_j} \sigma_{ij}}}. \tag{21}$$

As in the previous section, it can easily be seen that these coefficients (21) are always non-negative, which means that (20) is a comonotonic sum. We call $S_i^l(\underline{\pi})$ a *comonotonic lower bound*. Because of the additivity property of Theorem 1, it is straightforward to determine the distribution function and related risk measures of the approximations $S_i^l(\underline{\pi})$.

We know from e.g. Dhaene et al. (2005) that the comonotonic lower bound as given by (20) is a very accurate approximation of $S_i(\underline{\pi})$.

If rebalancing is performed several times per year, or $p > 1$, we can approximate each sum in (15) by a convex lower bound similar to (20). We will show further on that the resulting approximation $S_i^l(\underline{\pi})$ is again a convex lower bound for $S_i(\underline{\pi})$.

Using these results, we can accurately approximate $W_n^P(\underline{\pi})$ by:

$$W_n^{P,l}(\underline{\pi}) = \sum_{i=0}^{n} \alpha_i \prod_{j=i+1}^{n} S_j^l(\underline{\pi}) = \left[\left((\alpha_0 S_1^l(\underline{\pi}) + \alpha_1) S_2^l(\underline{\pi}) + \alpha_2\right) S_3^l(\underline{\pi}) + ...\right] S_n^l(\underline{\pi}) + \alpha_n. \tag{22}$$

We can use the following theorem to show that our approximation $W_n^{l,P}(\underline{\pi})$ is a convex lower bound for $W_n^P(\underline{\pi})$.

Theorem 2 *Given deterministic amounts $\alpha_0, \alpha_1, ..., \alpha_n$, and given two sequences of independent risks $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ such that $X_i \leq_{cx} Y_i$ holds for $i = 1, 2, ..., n$, we have that*

$$\sum_{i=0}^{n} \alpha_i \prod_{j=i+1}^{n} X_j \leq_{cx} \sum_{i=0}^{n} \alpha_i \prod_{j=i+1}^{n} Y_j \tag{23}$$

Proof. The proof of this result is similar to the proof of Proposition 3.4.25(ii) in Denuit et al. (2005). We will proof the result using induction. For $n = 1$, the result follows immediately. Now suppose $W_{n-1}^{l} \leq_{cx} W_{n-1}$. For every convex function v we have

$$\begin{aligned}
\mathrm{E}\left[v\left(\sum_{i=0}^{n} \alpha_i \prod_{j=i+1}^{n} X_j \right) \middle| \, X_n = x \right] &= \mathrm{E}\left[v\left(x\left(\sum_{i=0}^{n-1} \alpha_i \prod_{j=i+1}^{n-1} X_j \right) + \alpha_n \right) \right] \\
&\leq \mathrm{E}\left[v\left(x\left(\sum_{i=0}^{n-1} \alpha_i \prod_{j=i+1}^{n-1} Y_j \right) + \alpha_n \right) \right] \\
&= \mathrm{E}\left[v\left(X_n\left(\sum_{i=0}^{n-1} \alpha_i \prod_{j=i+1}^{n-1} Y_j \right) + \alpha_n \right) \middle| \, X_n = x \right].
\end{aligned}$$

Taking the expectation on both sides leads to

$$\mathrm{E}\left[v\left(\sum_{i=0}^{n} \alpha_i \prod_{j=i+1}^{n} X_j \right) \right] \leq \mathrm{E}\left[v\left(X_n \sum_{i=0}^{n-1} \alpha_i \prod_{j=i+1}^{n-1} Y_j + \alpha_n \right) \right].$$

Repeating the argument, but this time conditioning on $Y_2, ..., Y_n$ and using the induction hypothesis with $n = 1$, we see that

$$\mathrm{E}\left[v\left(\sum_{i=0}^{n} \alpha_i \prod_{j=i+1}^{n} X_j \right) \right] \leq \mathrm{E}\left[v\left(\sum_{i=0}^{n} \alpha_i \prod_{j=i+1}^{n} Y_j \right) \right],$$

which proves the result. ■

In case yearly rebalancing is used ($p = 1$), it follows immediately from the theorem that $W_n^{P,l}(\underline{\pi})$ is a convex lower bound for $W_n^{P}(\underline{\pi})$.

As a special case of Theorem 2, it holds that

$$\prod_{i=1}^{n} X_i \leq_{cx} \prod_{i=1}^{n} Y_i, \tag{24}$$

for two sequences of independent risks $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ such that $X_i \leq_{cx} Y_i$ for $i = 1, 2, ..., n$. This means that, in case $p > 1$, the approximation of $S_i(\underline{\pi})$ is a convex lower bound of $S_i(\underline{\pi})$, since this approximation is obtained by replacing each of the sums in (15) by a convex lower bound. Knowing this, it follows immediately from Theorem 2 that $W_n^{l,P}(\underline{\pi})$ is a convex lower bound for $W_n^{P}(\underline{\pi})$, also if rebalancing is performed several times per year.

As is apparent from the formulas above, continuous rebalancing is from a computational point of view much more attractive compared to the discrete case. As seen from (16), the terminal wealth in case of periodic rebalancing is a sum of products of the yearly returns. Moreover, (15) shows that each of these yearly returns is on itself a product of sums of dependent lognormal variables. Overall the terminal wealth becomes extremely cumbersome and time-consuming to compute, even after approximating the sums of lognormals by (20). In Appendix A, a numerical scheme to implement periodic rebalancing, and more precisely formula (22), is described.

On the other hand, constant mix strategies are much more convenient. As can be seen from (9), continuous rebalancing reduces the terminal wealth to a sum of (dependent) lognormal random variables, which, after approximating it by (11), is straightforward to implement. As a consequence, continuous rebalancing allows for much more general optimal portfolio selection problems to be solved, as can be found in e.g. Dhaene et al. (2005) and Van Weert et al. (2010a & 2010b).

In the following section, numerical examples are given to illustrate that constant mix strategies can be used safely instead of the more realistic class of periodic rebalancing strategies.

6 Numerical examples

In this section we provide numerical examples illustrating the appropriateness of continuous rebalancing, as explained in Section 4, as an approximation for the discrete case of Section 5. Example 1 indicates that constant mix strategies are the limiting case of periodically rebalanced investment strategies, as has been proven in Milevsky & Posner (1998). From examples 2 and 3 we see that results obtained by applying continuous rebalancing are very close to those obtained by yearly rebalancing. The numerical results in this section are obtained by applying the recursive scheme as described in Appendix A.

Example 1: One year return In this example we illustrate that, as stated in Section 4, increasing the number of times the investor rebalances his portfolio leads in the limit to the corresponding constant mix strategy. Suppose we have two asset classes available in which we can invest. The first asset class has drift and standard deviation equal to $(\mu_1, \sigma_1) = (0.03, 0.05)$. The second asset class is more risky, with parameters $(\mu_2, \sigma_2) = (0.06, 0.20)$. Assume that the correlation between the two asset classes equals $\rho_{12} = 0.5$. Furthermore, assume that the investor chooses to distribute his wealth equally between the two classes, or $\underline{\pi}^T = (0.5, 0.5)$.

Assume the investor saves $\alpha_0 = 1$ at time 0. We compute the distribution of the wealth available on the account at time 1. In Figure 1a, 1c and 1d, the dashed lines are obtained by periodic rebalancing, by approximately computing the yearly return as given by (15). From left to right, the dashed lines represent p = 1, 2, 3, 4, 5, 6 and 10 respectively. The full line in Figure 1 corresponds to the continuous case (11), with $n = 1$. Looking at the whole distribution function in Figure 1a, we can not distinguish the different lines. This is confirmed by the QQ-plot in Figure 1b, as the quantiles form a straight line. This QQ-plot compares the quantiles obtained by continuous and yearly rebalancing ($p = 1$). For $p > 1$, the difference with continuous rebalancing is even smaller. In Figures 1c and 1d respectively, the quantiles around 80% and 30% are depicted. We see that increasing the number of times the portfolio is rebalanced shifts the resulting distribution function closer to the continuous case. Moreover, we see that continuous rebalancing always outperforms periodic rebalancing, even though the difference is very small.

INSERT FIGURE 1 HERE

In Table 1, Appendix B, exact numerical values are given for some selected quantiles.

Example 2: Terminal wealth at time 10, single investment Suppose we have two asset classes available, with $(\mu_1, \sigma_1) = (0.05, 0.05)$ and $(\mu_2, \sigma_2) = (0.07, 0.10)$. Also suppose the investor saves a single amount of one at time zero over an investment horizon of 10 years, or $n = 10$, $\alpha_0 = 1$ and $\alpha_i = 0$ for $i > 0$. Assume that the correlation between the two asset classes equals $\rho_{12} = 0.5$, and that the investor invests according to $\underline{\pi}^T = (0.5, 0.5)$.

In this example we focus on the wealth available on the account at time 10. Applying formula (11) we can determine the distribution function of the wealth obtained by continuous rebalancing. This distribution is depicted by the full line in Figure 2. We compare this result to the wealth obtained through yearly rebalancing, which we compute using (22) and corresponds to the dashed line in Figure 2. Looking at Figure 2a and the QQ-plot in Figure 2b, we see that both distribution functions are very close to each other. If we zoom in, e.g. on the area around the 80%-quantile as in Figure 2c, or around the 40%-quantile as in Figure 2d, we come to the same conclusions as in the Example 1. First of all we see that continuous rebalancing slightly outperforms the periodic case. Furthermore, the figure indicates that continuous rebalancing serves as a very good approximation for periodic rebalancing, even when rebalancing is performed only once per year.

INSERT FIGURE 2 HERE

In Table 2, Appendix B, exact numerical values are given for some selected quantiles.

Example 3: Terminal wealth at time 10, periodic investments As a third and final example, assume we have the same asset classes available as in Example 2, and assume the investor invests his wealth again according to $\underline{\pi}^T = (0.5, 0.5)$. Assume now that the investor saves an amount of one each year for a period of 10 years, or $\alpha_i = 1$ for $0 \leq i \leq 9$. As in the previous example, we are interested in the available wealth at time 10. In Figure 3 the distribution function of the wealth at time 10 is depicted. Again, the full line corresponds to the distribution function of the wealth obtained by continuous rebalancing, which is computed using (11). The dashed line corresponds to yearly rebalancing, obtained by applying (22). We see once again that continuous rebalancing leads to a slightly bigger terminal wealth. But, more importantly, the difference between the two distributions functions is so small that we can conclude that a constant mix strategy can safely be used in practice as an approximation for a periodically rebalanced strategy, even when rebalancing is only performed on a yearly basis.

INSERT FIGURE 3 HERE

In Table 3, Appendix B, exact numerical values are given for some selected quantiles.

In the examples in this section, changing the assumptions (different or more asset classes, a different cash flow scheme and/or a different investment strategy) leads to the same conclusions.

7 Conclusion

The main purpose of this paper was to discuss the appropriateness of using a constant mix strategy as an approximation for a periodically rebalanced investment strategy. Overall, the examples of Section 6 confirm the theoretical finding of e.g. Milevsky & Posner (1998) that, as the time period between rebalancing decreases, periodically rebalanced strategies move towards the corresponding constant mix strategy. In other words, continuous rebalancing can be seen as a limiting case of periodic rebalancing. Secondly, we can say that a constant mix strategy in general outperforms its corresponding periodically rebalanced strategy.

The final, and most significant conclusion from our numerical examples is that results obtained using constant mix strategies are very close to those obtained by periodic rebalancing. This is important because, as indicated in Sections 4 and 5, constant mix strategies have clear computational advantages. As shown e.g. in Dhaene et al. (2005), convex order bounds based on comonotonicity are very useful to solve very general optimization problems. However, Dhaene et al. restrict their results to constant mix strategies, which can not be applied in practice, since they require rebalancing of the investments on a continuous basis. In practice, an investor can only rebalance his investment portfolio on discrete points in time. However, when solving general optimization problems, such as determining the investment strategy leading to a maximal terminal wealth, using discrete rebalancing is unfeasible, even when applying comonotonic approximations. The results from this paper justify, even though it is not realistic in practice, the use of constant mix strategies to approximate periodically rebalanced strategies, allowing much more general optimization problems to be solved.

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A Periodic rebalancing: recursive scheme

In this appendix we give a description of the recursive numerical scheme we used to compute the terminal wealth for the class of periodic rebalancing investment strategies. We restrict to the special case of yearly rebalancing ($p = 1$), but the scheme can be extended easily to shorter time periods between rebalancing ($p > 1$).

For a given investment strategy $\underline{\pi}$, we want to compute the distribution function of $W_n^{P,l}(\underline{\pi})$, as given by (22). We know that at time 0, $W_0^{P,l}(\underline{\pi})$ is simply equal to α_0. Suppose we have computed the distribution of $W_j^{P,l}(\underline{\pi})$ for some $1 \leq j < n$. As seen from (22), the (approximate) wealth at time $j+1$ is then given by

$$W_{j+1}^{P,l}(\underline{\pi}) = W_j^{P,l}(\underline{\pi}) S_{j+1}^l(\underline{\pi}) + \alpha_{j+1} \tag{25}$$

Hereafter, for notational convenience, we omit the dependence of our variables on the investment strategy $\underline{\pi}$.

The algorithm to compute the distribution function of $W_{j+1}^{P,l}$ starts by the discretization of the logarithm of the continuous random variables $W_j^{P,l}$ and S_{j+1}^l. More information on this discretization scheme can be found in the appendix of Klugman et al. (2008). Next, these discretized variables are convoluted using the discrete Fourier Transform. Finally, we get the desired values for $W_{j+1}^{P,l}$ by applying undiscretization, see again Klugman et al. (2008).

This results in the following scheme:

- Determine a suitable $x_{\min}$ and $x_{\max}$ such that $F_{W_j^{P,l}}(e^{x_{\min}}) = F_{S_{j+1}^l}(e^{x_{\min}}) = 0$ and $F_{W_j^{P,l}}(e^{x_{\max}}) = F_{S_{j+1}^l}(e^{x_{\max}}) = 1$. Choose the number of steps n big enough, and define the span $k = \frac{x_{\min} - x_{\max}}{n}$.
- Compute $F_{\log W_j^{P,l}}(x_i) = F_{W_j^{P,l}}(e^{x_i})$ in the points $x_i = x_0 + ik$, for $i = 0,..., n$.
- Using (20), compute $F_{\log S_{j+1}^l}(x_i) = F_{S_{j+1}^l}(e^{x_i})$ in the points $x_i = x_0 + ik$, for $i = 0,...,$ n.
- Define $y_i = x_i + \frac{k}{2}$. Determine the discretized version of the continuous random variable $\log W_j^{P,l}$, which we denote as $\left(\log W_j^{P,l}\right)_d$, in the points y_i as follows:

$$f_{\left(\log W_j^{P,l}\right)_d}(y_i) = \Pr\left[x_i \leq \log W_j^{P,l} \leq x_{i+1}\right] = F_{\log W_j^{P,l}}(x_{i+1}) - F_{\log W_j^{P,l}}(x_i),\ i = 0,...,\ n-1 \tag{26}$$

We have that

$$f_{\left(\log W_j^{P,l}\right)_d}(y_i) = f_{Y_1}(i)\,, \tag{27}$$

with the random variable Y_1 given by $Y_1 = \frac{1}{k}\left(\left(\log W_j^{P,l}\right)_d - x_{\min} - \frac{k}{2}\right)$.

- Determine the discretized version of $\log S_{j+1}^l$, denoted as $\left(\log S_{j+1}^l\right)_d$, in the points y_i as:

$$f_{\left(\log S_{j+1}^l\right)_d}(y_i) = F_{\log S_{j+1}^l}(x_{i+1}) - F_{\log S_{j+1}^l}(x_i)\,,\; i = 0,..., n \tag{28}$$

We have that

$$f_{\left(\log S_{j+1}^l\right)_d}(y_i) = f_{Y_2}(i)\,, \tag{29}$$

with the random variable Y_2 given by $Y_2 = \frac{1}{k}\left(\left(\log S_{j+1}^l\right)_d - x_{\min} - \frac{k}{2}\right)$.

- Using Fast Fourier transform techniques, we can compute the convolution of Y_1 and Y_2. Denoting the discrete Fourier Transform as $\mathcal{F}$, it holds that

$$f_{Y_1+Y_2} = \mathcal{F}^{-1}\left(\mathcal{F}(Y_1)\,\mathcal{F}(Y_2)\right) \tag{30}$$

This allows us to compute values for $f_{Y_1+Y_2}(i)$, for $i = 0,..., n-1$. We have that

$$f_{Y_1+Y_2}(i) = f_{\left(\log W_j^{P,l}\right)_d + \left(\log S_{j+1}^l\right)_d}(z_i)\,, \tag{31}$$

with $z_i = ki + 2x_{\min} + k$.

- Using undiscretization we can get values for

$$F_{\log W_j^{P,l} + \log S_{j+1}^l}(x) = F_{W_j^{P,l} S_{j+1}^l}(e^x) \tag{32}$$

for any x as follows:

$$F_{\log W_j^{P,l} + \log S_{j+1}^l}(x) = \sum_{t=0}^{i-1} f_{\left(\log W_j^{P,l}\right)_d + \left(\log S_{j+1}^l\right)_d}(z_t) + \frac{x - z_i}{k} f_{\left(\log W_j^{P,l}\right)_d + \left(\log S_{j+1}^l\right)_d}(z_i)\,, \tag{33}$$

with i such that $z_i \le x < z_{i+1}$.

B Numerical results

In this appendix we give the numerical results corresponding to the examples in Section 6. For each example we give the numerical results for a region of selected probability levels. In the tables the relative difference is also given, which is calculated as

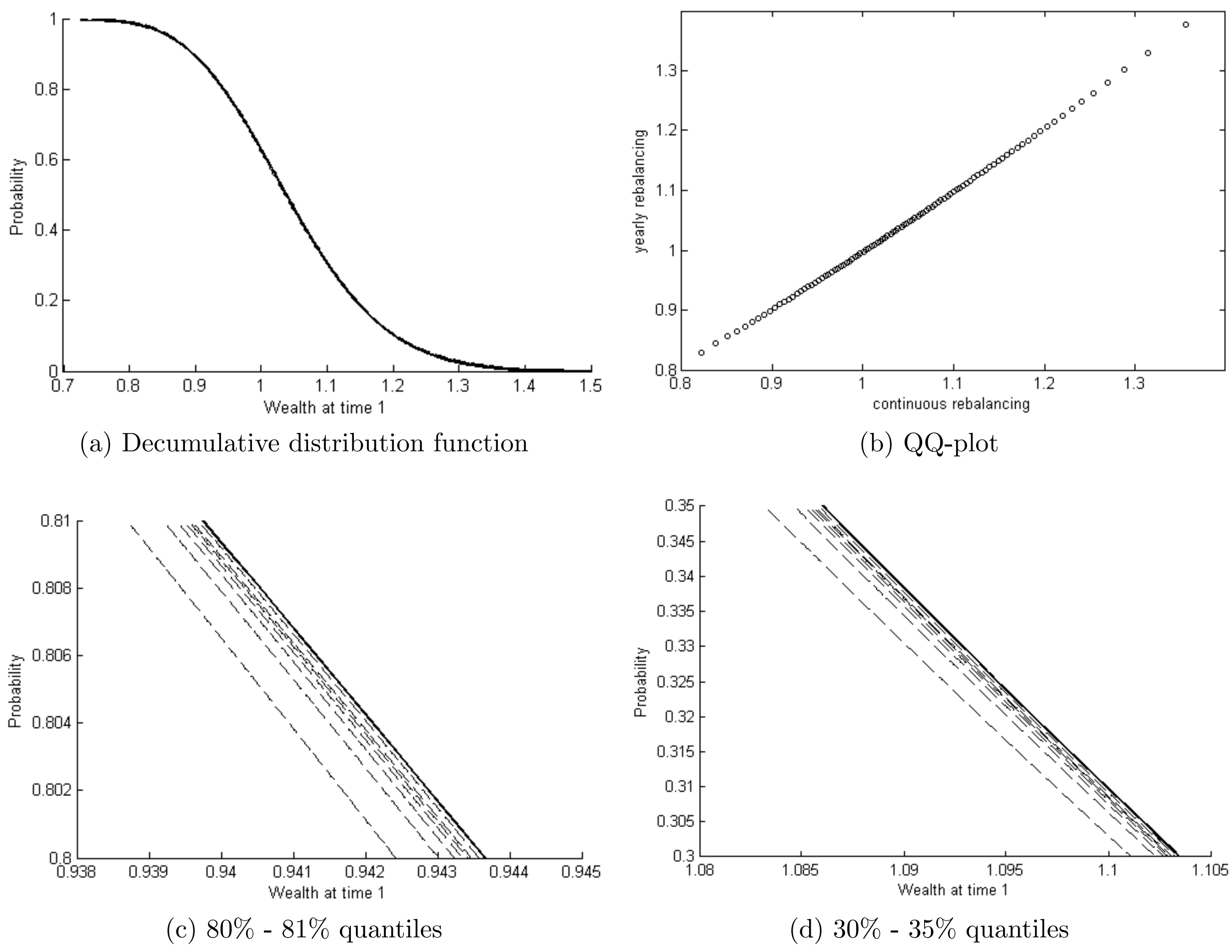
$$\frac{W_n^{C,l}(\underline{\pi}) - W_n^{P,l}(\underline{\pi})}{W_n^{C,l}(\underline{\pi})}. \tag{34}$$

The last column in Table 1 corresponds to the relative difference between results obtained by yearly rebalancing ($p = 1$) and continuous rebalancing.

INSERT TABLE 1 HERE

INSERT TABLE 2 HERE

INSERT TABLE 3 HERE



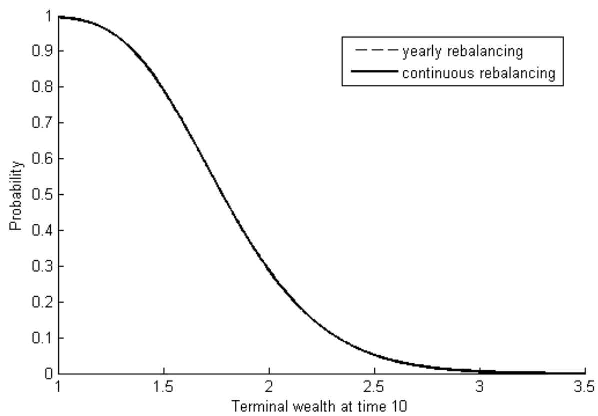
(a) Decumulative distribution function

(b) QQ-plot

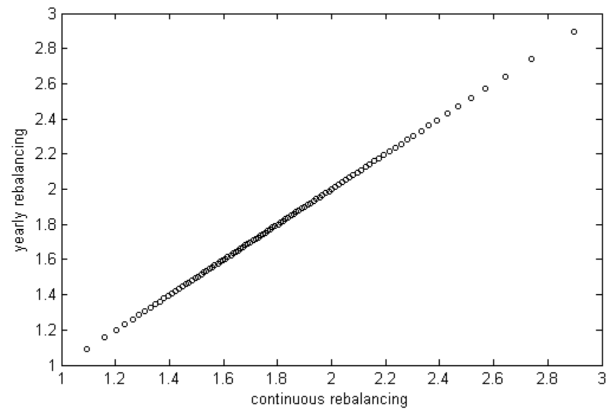
(c) 80% - 81% quantiles

(d) 30% - 35% quantiles

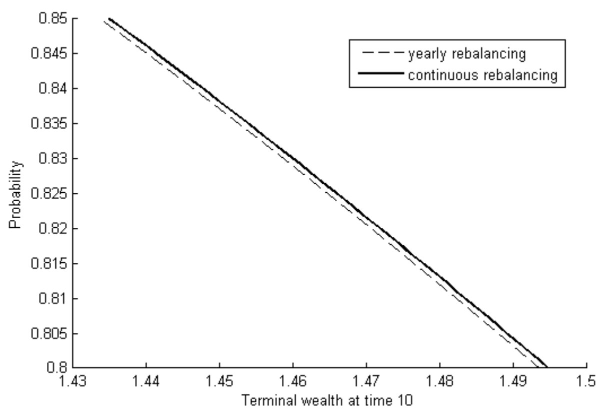
Figure 1: Wealth at time 1: continuous vs. discrete rebalancing ($\mu^T = [0.03, 0.06]$, $\sigma^T = [0.05, 0.2]$, $\rho_{12} = 0.5$, $\pi^T = [0.5, 0.5]$.



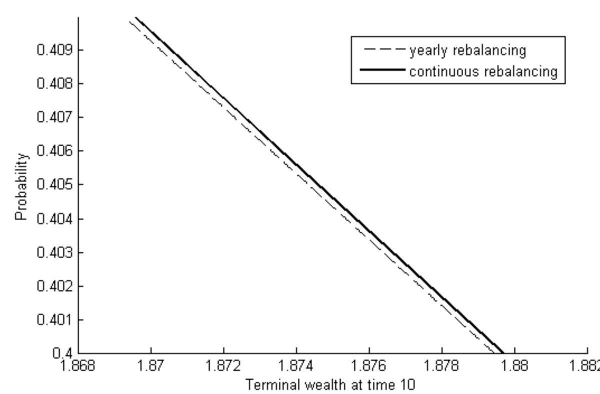
(a) Decumulative distribution function



(b) QQ-plot

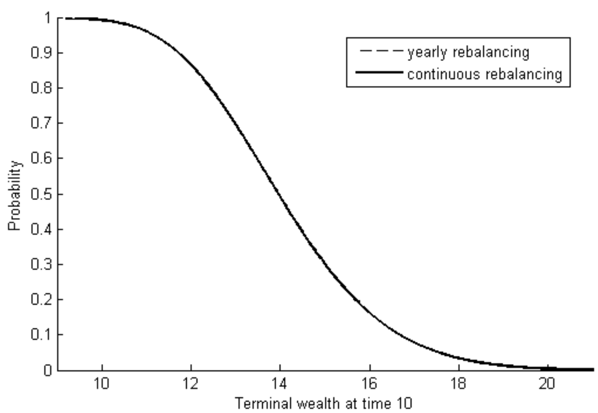


(c) 80% - 85% quantiles

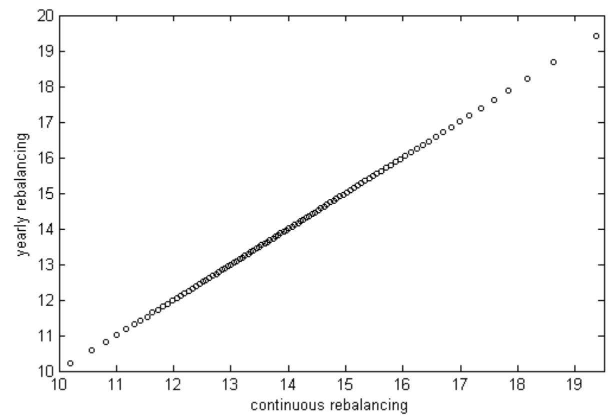


(d) 40% - 41% quantiles

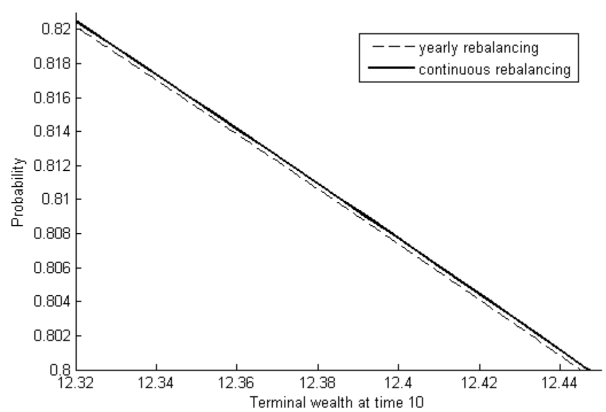
Figure 2: Terminal wealth at time 10: continuous vs. yearly rebalancing ($\mu^T = [0.05, 0.07]$, $\sigma^T = [0.05, 0.1]$, $\rho_{12} = 0.5$, $\pi^T = [0.5, 0.5]$, $\alpha_0 = 1$, $\alpha_i = 0$ for $0 < i \leq 9$.



(a) Decumulative distribution function



(b) QQ-plot



(c) 80% - 82% quantiles

Figure 3: Terminal wealth at time 10: continuous vs. yearly rebalancing ($\mu^T = [0.05, 0.07]$, $\sigma^T = [0.05, 0.1]$, $\rho_{12} = 0.5$, $\pi^T = [0.5, 0.5]$, $\alpha_i = 1$ for $0 \leq i \leq 9$.

probability	p							continuous	relative
level	1	2	3	4	5	6	10	rebalancing	difference
0.05	1.2545	1.2554	1.2559	1.2562	1.2566	1.2573	1.2586	1.2628	0.65%
0.10	1.2034	1.2038	1.2040	1.2042	1.2043	1.2046	1.2051	1.2068	0.28%
0.15	1.1701	1.1703	1.1703	1.1704	1.1704	1.1705	1.1706	1.1710	0.08%
0.20	1.1437	1.1440	1.1441	1.1442	1.1442	1.1442	1.1443	1.1443	0.06%
0.25	1.1210	1.1218	1.1221	1.1222	1.1223	1.1224	1.1225	1.1226	0.15%
0.30	1.1011	1.1023	1.1027	1.1029	1.1030	1.1031	1.1033	1.1035	0.22%
0.35	1.0832	1.0846	1.0851	1.0854	1.0855	1.0856	1.0858	1.0861	0.27%
0.40	1.0666	1.0682	1.0687	1.0690	1.0691	1.0692	1.0695	1.0698	0.30%
0.45	1.0508	1.0525	1.0531	1.0534	1.0536	1.0537	1.0539	1.0542	0.32%
0.50	1.0357	1.0374	1.0380	1.0383	1.0385	1.0386	1.0389	1.0392	0.33%
0.55	1.0209	1.0226	1.0232	1.0235	1.0236	1.0238	1.0240	1.0243	0.33%
0.60	1.0062	1.0078	1.0083	1.0086	1.0088	1.0089	1.0091	1.0094	0.32%
0.65	0.9913	0.9928	0.9933	0.9935	0.9937	0.9938	0.9940	0.9943	0.30%
0.70	0.9760	0.9772	0.9777	0.9779	0.9780	0.9781	0.9783	0.9786	0.26%
0.75	0.9599	0.9608	0.9612	0.9614	0.9615	0.9615	0.9617	0.9619	0.21%
0.80	0.9424	0.9430	0.9432	0.9433	0.9434	0.9434	0.9435	0.9436	0.13%
0.85	0.9226	0.9227	0.9227	0.9227	0.9228	0.9228	0.9228	0.9228	0.02%
0.90	0.8972	0.8974	0.8975	0.8975	0.8976	0.8977	0.8979	0.8987	0.16%
0.95	0.8606	0.8611	0.8613	0.8615	0.8617	0.8620	0.8627	0.8650	0.50%

Table 1: Selected quantiles: Example 1

probability level	continuous rebalancing	yearly rebalancing	relative difference
0.05	2.5165	2.5142	0.088%
0.10	2.3323	2.3305	0.079%
0.15	2.2153	2.2140	0.058%
0.20	2.1265	2.1257	0.039%
0.25	2.0531	2.0527	0.024%
0.30	1.9894	1.9892	0.010%
0.35	1.9322	1.9321	0.004%
0.40	1.8796	1.8793	0.016%
0.45	1.8301	1.8296	0.026%
0.50	1.7826	1.7820	0.035%
0.55	1.7364	1.7356	0.044%
0.60	1.6906	1.6897	0.053%
0.65	1.6446	1.6436	0.060%
0.70	1.5974	1.5963	0.068%
0.75	1.5480	1.5469	0.075%
0.80	1.4948	1.4936	0.084%
0.85	1.4351	1.4338	0.095%
0.90	1.3634	1.3619	0.110%
0.95	1.2635	1.2615	0.155%

Table 2: Selected quantiles: Example 2

probability level	continuous rebalancing	yearly rebalancing	relative difference
0.05	17.6066	17.5763	0.172%
0.10	16.7170	16.6970	0.120%
0.15	16.1452	16.1313	0.087%
0.20	15.7073	15.6969	0.066%
0.25	15.3417	15.3346	0.046%
0.30	15.0221	15.0173	0.032%
0.35	14.7335	14.7299	0.025%
0.40	14.4649	14.4628	0.014%
0.45	14.2104	14.2096	0.006%
0.50	13.9653	13.9653	0.000%
0.55	13.7258	13.7251	0.005%
0.60	13.4871	13.4856	0.011%
0.65	13.2453	13.2431	0.017%
0.70	12.9959	12.9938	0.016%
0.75	12.7326	12.7302	0.019%
0.80	12.4466	12.4438	0.022%
0.85	12.1224	12.1202	0.018%
0.90	11.7279	11.7262	0.015%
0.95	11.1698	11.1694	0.004%

Table 3: Selected quantiles: Example 3