Decision principles derived from risk measures

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Abstract

In this paper, we argue that a distinction exists between risk measures and decision principles. Though both are functionals assigning a real number to a random variable, we think there is a hierarchy between the two concepts. Risk measures operate on the first "level", quantifying the risk in the situation under consideration, while decision principles operate on the second "level", often being derived from the risk measure. We illustrate this distinction with several canonical examples of economic situations encountered in insurance and finance.

Special attention is paid to the role of axiomatic characterizations in determining risk measures and decision principles. Some new axiomatic characterizations of families of risk measures and decision principles are also presented.

Keywords: Risk measurement; Decision-making; Axiomatization; Measure of risk; Premium principles; Solvency capital principles; Risk transfer principles; Equivalent utility; Esscher transform; Exponential utility.

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1 Introduction

We distinguish between *risk measures* on the one hand and *decision principles*, such as premium principles, solvency capital principles and risk transfer principles, on the other. The difference between risk measures and decision principles comes from the different "levels" on which they operate; that is, there is a hierarchy between the two concepts.

A *risk measure* is a functional that assigns a real number to a random variable (the risk). Justifications of risk measures are generally based on axiomatic characterizations. The general purpose of an axiomatic characterization is to demonstrate what are the essential assumptions to be imposed and what are relevant concepts (functions, parameters, et cetera) to be determined. A risk measure is appropriate if and only if its characterizing axioms are. Axiomatic characterizations can be used to justify a risk measure, but also to criticize it (see for example Wakker, 2004, and Laeven, 2005, Chapter 1).

The particular set of axioms must reflect the risk perception of the economic *actors* (or *agents*) involved in the situation under consideration. The economic relevance of the axioms thus depends on the actors involved as well as on the specific situation under study. The axioms should be formalized such as to be representative for all the actors involved in the evaluation of any "feasible" risk.

A *decision principle* is often a "derived" functional, assigning again a real number to a random variable. The derivation is typically based on an optimization procedure, for example by minimizing the total risk as measured by a risk measure, or on an equilibrium criterion. Notice that both risk measures and decision principles are functionals mapping random variables to the real line. Hence, mathematically they are similar concepts. However, justifications and derivations differ.

The explicit distinction between risk measures and decision principles enhances our understanding of risk evaluation mechanisms and the resulting decisions taken. By using different sets of axioms, reflecting different perceptions of risk, for example for different actors (management, regulator), or for different economic situations, appropriate decision principles can be derived.

Let us quote Markowitz (1959, Chapter 10) in this respect: "We might decide that in one context one set of basic principles is appropriate, while in another a different set of principles should be used. We might find that some patterns of preferences are consistent with a set of principles while other patterns are not." Here, "principles" mean "axioms".

We think that a "two level" procedure in which risk measures determine the risk of an economic transaction, and decision principles are derived in a second stage, is a proper approach and a valuable tool to understand what essentially drives the mechanisms of risk evaluation used and the decision principles following.

In the approach taken in this paper, the "selection" of appropriate preference axioms takes place on the highest "level" of measuring the total risk of an economic operation, while properties of decision principles follow as consequences.

In actuarial science, for many years, risk measures and decision principles—without explicit distinction—have been important objects of study; see for example Bühlmann (1970), Gerber (1979) and Goovaerts, De Vylder & Haezendonck (1984) for early accounts.

Recent developments aiming for international convergence of solvency capital principles, both in insurance and finance, have further increased the importance of the topic. In this context, we refer the interested reader to, for example the Basel and (current and upcoming) EU Solvency Capital Accords.

Framework and notation

In what follows, we will elucidate the concepts of risk measures and decision principles in prototypical economic situations, encountered mainly in insurance but also in finance. We assume that in the background, there is a measurable space $(\Omega, \mathcal{F})$ where Ω is the outcome space and $\mathcal{F}$ is a (σ-)algebra defined on it. A risk or random variable (r.v.) is an $\mathcal{F}$-measurable function defined on $(\Omega, \mathcal{F})$. We do not fix a probability measure on $\mathcal{F}$, i.e., we do not assume that a probability measure is given beforehand on $\mathcal{F}$; however, we tacitly assume that the measurable space $(\Omega, \mathcal{F})$ is always equipped with (at least) one probability measure, so that the $\mathcal{F}$-measurable functions are truly random variables. A risk represents the final net loss of a position (contingency) currently held. When $X > 0$, we call it a loss, whereas when $X \leq 0$, we call it a gain. The class of all r.v.'s on $(\Omega, \mathcal{F})$ is denoted by $\mathcal{X}$. Furthermore, we let $\mathcal{P}$ be the class of all probability measures on $(\Omega, \mathcal{F})$.

Mathematically, a *risk measure* π, or a *decision principle* ρ, is a functional assigning a real number to any r.v. defined on $(\Omega, \mathcal{F})$; that is, π and ρ are mappings from $\mathcal{X}$ to $\mathbb{R}$. Statements and definitions provided below hold for all $X, Y \in \mathcal{X}$ unless mentioned otherwise.

In the classical case, it is assumed that risks with identical distribution functions lead to the same value of the risk measure, that is, if for two risks X, Y we have that $\mathbb{P}[X \leq x] =: F_X(x) = F_Y(x) := \mathbb{P}[Y \leq x]$ for all real x and a given (reference) probability measure $\mathbb{P} \in \mathcal{P}$, then $\pi[X] = \pi[Y]$. This assumption is known as ($\mathbb{P}$-)*law invariance* (or independence, or objectivity). In general this assumption need not be imposed. We note that, to have equal distributions, the r.v.'s X and Y need not be defined on the same measurable space.

Though the units of the elements of $\mathcal{X}$ (the risks) are considered to be monetary units (for example euros or dollars), the units of π are not necessarily monetary. One can transform monetary units into dimensionless quantities, for example by considering ratios such as X/u, expressing the risk in proportion to the available capital u. Risk measures and decision principles for variables thus transformed have the advantage that they might be currency independent, even if not positively homogeneous. Consider for example the exponential premium $\rho_\alpha[X] = \frac{1}{\alpha}\log\mathbb{E}[e^{\alpha X}]$. Then a change of currency $X \to cX$ results in

$$\rho_\alpha[cX] = c\left[\frac{1}{c\alpha}\log\mathbb{E}[e^{c\alpha X}]\right] = c\rho_{\alpha c}[X].$$

Hence even though the exponential premiums are not positively homogeneous (scale invariant, scale equivariant) because $\rho[aX] = a\rho[X]$ for all $a \geq 0$ and all $X \in \mathcal{X}$ is not satisfied, the premium resulting is still correct if we only adapt the exponential parameter (coefficient of absolute risk aversion), which has dimension $(\text{money})^{-1}$.

It should be noted that the functional form of some decision principles derived in this paper (being mappings from $\mathcal{X}$ to $\mathbb{R}$) can also be axiomatized directly. In such cases, the risk measure and decision principle coincide. A direct axiomatic characterization need not be inferior as long as the axioms are well-chosen and undeniable. However, care should be taken not to mix up preference axioms with respect to risk evaluation, and axioms (or properties) for decision principles.

1.1 A note on the literature

The academic research on risk measures has recently experienced a revival. As is well-known, the study of risk measures and decision principles has a long history in actuarial science and probability theory. Furthermore, functionals representing preferences have been the object of study of microeconomic theory, in particular the realm of decision under uncertainty, for almost a century.

It is inevitable that a revival leads to the restatement (or reinvention) of known results, perhaps in a slightly different framework. Hereby, sometimes the source of the known results is properly cited, sometimes this is unfortunately omitted.

Examples of such restatements are listed below (without being exhaustive). We hope that this short list will enhance careful citation in future research.

- *Distortion risk measures* were developed in the microeconomics literature; see Schmeidler (1986) and Schmeidler (1989; the first version of this paper dates from 1982), Quiggin (1982) and Yaari (1987). In fact, an axiomatic characterization of the Choquet expectation (of which a distortion risk measure is a particular example) was already in Greco (1982) (see Denneberg, 1994, for a translation of the main results into English); see also Theorem 3 of the early Anger (1977).

- An axiomatic characterization of the upper (lower) expectation (also known as *coherent* risk measure) was established by Huber (1981). Related results in an economic environment are in Gilboa & Schmeidler (1989), providing an axiomatic foundation of the early Wald (1950).
- *Convex risk measures* were studied—but not axiomatized—already in Deprez & Gerber (1985). Axiomatic characterizations are due to Heath (2000), Föllmer & Schied (2002), Fritelli & Rosazza Gianin (2002) and Heath & Ku (2004). Precise connections between convex risk measures and the decision theory of Gilboa & Schmeidler (1989) were obtained recently by Laeven & Stadje (2010).
- Worst case risk measurement was studied in detail already in the operations research literature (see for example Ben-Israel, Ben-Tal & Zlobec, 1981, and the references therein) and in actuarial science by De Vylder (1982, and subsequent papers); see also Section 7 of this paper for further details.

2 Premium principles

A prime example of a decision principle in an insurance context is an insurance premium principle. When deriving a premium principle, two viewpoints can be taken, the one of the insurer and the one of the insured.

From the viewpoint of the insurer, a premium principle can for example be derived such that the probability of ruin is sufficiently small (see for example Gerber, 1974, Bühlmann, 1985, or Kaas *et al.*, 2008, Section 5.2, for further details in this direction). Another well-known approach, following the axiomatization of Von Neumann & Morgenstern (1944), is to specify a non-decreasing function $u : \mathbb{R} \to \mathbb{R}$, referred to as a *utility* function, and consider the expected utility $\mathbb{E}[u(w+\rho[X]-X)]$, where w denotes the initial wealth of the insurer, X is the loss incurred due to the insured risk during the period considered and ρ denotes the premium principle. In the Von Neumann-Morgenstern framework, the utility function u is subjective, whereas the probability measure is assumed to be objective, that is, known and given beforehand. In the more general framework of Savage's (1954) axiomatization, which we adopt here, also the probability measure can be subjective: in the latter framework, the probability measure, just like the utility function u, may be based on subjective judgements of the decision situation under consideration.

From the expected utility expression based on Savage's numerical representation, an *equivalent utility principle* (also known under the slight misnomer *zero utility principle*) can be established as follows (see for example Bowers *et al.*, 1997, pp. 9-10, Denuit *et al.*, 2006, and Laeven & Goovaerts, 2008): for a given r.v. X and a given real number w representing the wealth of the company without the new contract, the equivalent utility premium $\rho^-[X]$ of the insurer is derived by solving the equation

$$\mathbb{E}[u(w + \rho[X] - X)] = u(w), \tag{1}$$

where the expectation is computed with respect to the subjective probability measure. Under the assumption that u is continuous (which is implied by the usual set of axioms characterizing expected utility preferences) and provided that the expectation is finite, a solution exists. If the insurer is risk-averse (which for expected utility preferences is equivalent to the utility function being concave), one easily proves that $\rho^-[X] \geq \mathbb{E}[X]$. The insurer will sell the insurance if and only if he can charge a premium $\rho[X]$ that satisfies $\rho[X] \geq \rho^-[X]$.

Next, we consider the viewpoint of the insured. An insurance treaty that for a given r.v. X leaves the insured with final wealth $\bar{w} - \rho[X]$ will be preferred to full self insurance, which leaves the insured with final wealth $\bar{w} - X$, if and only if $\bar{u}(\bar{w} - \rho[X]) \geq \mathbb{E}[\bar{u}(\bar{w} - X)]$, where $\bar{u}$ denotes the utility function of the insured. The equivalent utility premium $\rho^+[X]$ is derived by solving the equivalence condition

$$\bar{u}(\bar{w} - \rho[X]) = \mathbb{E}[\bar{u}(\bar{w} - X)].$$

The insured will buy the insurance if and only if $\rho[X] \leq \rho^+[X]$. One easily verifies that a risk-averse insured is willing to pay more than the pure net premium $\mathbb{E}[X]$. An insurance treaty can be signed both by the insurer and by the insured only if the premium satisfies $\rho^-[X] \leq \rho[X] \leq \rho^+[X]$.

Notice that the properties of an equivalent utility premium are determined by the properties of the utility function; within the expected utility framework, assuming the utility function to be twice differentiable, the absolute risk aversion $-\frac{u''(x)}{u'(x)}$ captures all information relevant to the utility function. That is, the properties that the decision principle satisfies are determined by the risk measure it is derived from.

It is well-known that if the utility function is non-decreasing, which is implied by the monotonicity axiom that is imposed to axiomatize expected utility, then $X \leq_{\mathrm{st}} Y$ implies both $\rho^+[X] \leq \rho^+[Y]$ and $\rho^-[X] \leq \rho^-[Y]$. If moreover the expected utility maximizer is risk-averse, which is equivalent to concavity of the utility function, then $X \leq_{\mathrm{SL}} Y$ implies both $\rho^+[X] \leq \rho^+[Y]$ and $\rho^-[X] \leq \rho^-[Y]$. Here "$\leq_{\mathrm{st}}$" denotes smaller in stochastic order (larger cdf), while "$\leq_{\mathrm{SL}}$" denotes smaller in stop-loss order. See for example Kaas *et al.* (2008, Ch. 7).

While expected utility theory is the dominant *normative* theory for decision under uncertainty, other decision theories exist that often have better *descriptive* power. By application of the equivalent utility principle such theories generate alternative premium principles.

It is important to note that *all* premium principles derived from the equivalent utility principle under *all* decision theories considered are translation invariant (translation equivariant, translative, consistent), meaning that $\rho[X + b] = \rho[X] + b$ for all real b (see Denuit *et al.*, 2006). But note that translation invariance may be relevant only when the risk measure is used to assess the size of the risk (like the mean does), not its riskiness (like the standard deviation).

Observe that a "two level" procedure, in which decision principles (second level) emerge from the joint applying of risk measures (first level) and an equilibrium (equivalence) criterion, has a long (implicit) tradition in the premium principles literature.

An interesting problem, related to premium calculation as described above, is the question of how to distribute the aggregate premium income of an insurer between individual policies. Taking into account the specific nature of insurance pricing, the mechanism that distributes the aggregate premium income can to some extent be rather arbitrary, because the premium of an individual policy reflects two types of randomness: on the one hand, it reflects the randomness of the single r.v. (the risk) under consideration and on the other hand it reflects the randomness of the aggregate insurance portfolio, as well as its heterogeneity and the subsidizing that takes place within the portfolio.

For the case of mutually independent policies, Bühlmann (1985) proposes to attribute to policy j, $j = 1, \ldots, n$ with n denoting the total number of policies, the premium

$$\rho[X_j] = \frac{1}{\alpha} \log \mathbb{E}[\exp(\alpha X_j)], \tag{2}$$

which can be regarded as the equivalent utility premium of an insurer with an exponential utility function with constant absolute risk aversion α. It is well-known that (2) is additive for independent r.v.'s. In Bühlmann (1985), the allocation (2) arises not as an equivalent utility premium but rather in the framework of ruin probability theory when the probability of ruin is bounded from above by ε and the insurer has an exponential utility function. In this case, it turns out that α takes the value of $\frac{|\log \varepsilon|}{w}$, with w denoting the initial wealth of the insurer.

Goovaerts *et al.* (2004b) present a new and general axiomatic characterization of risk measures that are additive for independent r.v.'s. The risk measure characterized there is of the same form as the premium in expression (2), though mixed with respect to the parameter α. The interested reader is referred to Laeven & Goovaerts (2008), for a survey of premium principles, including approaches left undiscussed here.

3 Solvency capital principles

In the insurance and financial industry, solvency capital serves as a buffer against the contingency that assets turn out to be insufficient to cover current as well as future liabilities. Several types of solvency capital need to be distinguished, namely regulatory capital, economic (or management) capital, rating capital and book capital. While the discussion below focuses on economic capital, most of it also applies to other forms of solvency capital; for further details we refer to Laeven & Goovaerts (2004), Goovaerts, Van den Borre & Laeven (2005) and Dhaene *et al.* (2008).

In the insurance and financial industry, the inverse cdf $F_X^{-1}(p)$ is usually referred to as the *Value-at-Risk* of X at probability level p. We also will use the *Tail-Value-at-Risk* at probability level p, denoted by $\mathrm{TVaR}_p[X]$ and, for continuous cdfs F_X, equal to the Conditional Tail Expectation $\mathbb{E}\big[X \mid X > \mathrm{VaR}_p[X]\big]$. A useful interpretation is that the TVaR is in fact the area between the curves $y = \max\{p, F_X(x)\}$ and $y = 1$. See Kaas *et al.* (2008, Sec. 5.6 as well as Fig 5.1) for more details.

If $X \sim N(\mu, \sigma^2)$, then

$$F_X^{-1}(p) = \mu + \sigma\Phi^{-1}(p), \qquad \text{and} \qquad \mathrm{TVaR}_p[X] = \mu + \sigma\frac{\varphi(\Phi^{-1}(p))}{1-p},$$

with φ and Φ the pdf and cdf of a standard normal r.v., respectively.

We argue that the optimal amount of economic capital should be derived in a tradeoff between risk exposure on the one hand and the cost of economic capital on the other hand. A quite similar tradeoff is encountered in mathematical statistics, where hypothesis testing is not a problem with one criterion but a problem in which the probabilities of two possible errors are to be weighed: the so-called *type I error* of rejecting a true null hypothesis and the *type II error* of failing to reject a false null hypothesis.

Let ρ denote an economic capital principle. Trivially,

$$X \equiv \min(X, \rho[X]) + (X - \rho[X])_+.$$

In the insurance and financial industry, the "lower layer" $\min(X, \rho[X])$ can be regarded as the risk borne by the shareholders. When $X \leq 0$ (hopefully the usual case), a profit is made (and typically a dividend is paid out). When $X > 0$, the shareholders experience a loss. Because the loss experienced by the shareholders will not exceed the total amount of solvency capital invested, the loss is capped at $\rho[X]$. The risk $\min(X, \rho[X])$ will be evaluated by the shareholders and an opportunity cost of capital will be charged.

On the one hand, the larger the solvency capital to be committed by the shareholders, the larger the costs associated with it. On the other hand, the smaller the amount of solvency capital, the smaller the premium the insured, still being exposed to the residual risk $(X - \rho[X])_+$, is willing to pay.

Let $0 \leq \delta < 1$ denote the opportunity cost per unit of capital (in excess of the risk-free rate of interest) and let $\nu : \mathcal{X} \to \mathbb{R}$ be a valuation principle for the risk residual (ν can for example be obtained from an equivalent utility principle). Then we consider

$$\delta\rho[X] + \nu\left[(X - \rho[X])_+\right]. \tag{3}$$

The objective function (3), to be minimized with respect to the amount of economic capital $\rho[X]$, strikes a balance between shareholders' cost of capital $\delta\rho[X]$ on the one hand and policyholders' residual risk exposure $(X - \rho[X])_+$ on the other. It can be viewed as a middle ground between an axiomatic approach and a full-fledged structural model. For simplicity, we assume δ to be constant; in a more sophisticated structural model it should be endogenously determined.

One easily verifies (see Laeven & Goovaerts, 2004) that if ν is the expectation operator with respect to any probability measure (note the generality of the following result), the optimal solvency capital principle that minimizes (3) provided the expectation exists, is given by

$$\rho[X] = F_X^{-1}(1 - \delta).$$

It means that the optimal solvency capital principle is a Value-at-Risk at probability level $1 - \delta$. Furthermore, at the optimum, the objective function satisfies

$$\begin{aligned}\delta\rho[X] + \mathbb{E}\left[(X - \rho[X])_+\right] &= \delta F_X^{-1}(1 - \delta) + \mathbb{E}\left[(X - F_X^{-1}(1 - \delta))_+\right] \\ &= \delta \mathrm{TVaR}_{1-\delta}[X].\end{aligned}$$

The final equality used follows, for example, from (5.42) in Kaas *et al.* (2008).

We note that, with constant δ, the objective function (3) always gives rise to a translation invariant solvency capital principle. Furthermore, with constant δ, the solvency capital principle is positively homogeneous whenever ν is positively homogeneous. In reality, when the size of the risk increases substantially, the assumption of constant δ needs to be revisited due to a required liquidity premium.

4 Risk transfer principles

4.1 Optimal risk sharing

A cooperating "pool" with n participating insurance companies wants to insure a risk X. We assume that participant i has an exponential(α_i) utility function. The claim amount this participant has to pay is denoted by X_i, hence $X \equiv X_1 + \cdots + X_n$. To compete optimally, the pool seeks to find the claim distribution (allocation) $(X_1, \ldots, X_n)$ for which the total premium income needed to keep each participant's utility at the same level is minimal. That is, the pool looks for

$$\inf_{(X_1,\ldots,X_n)\,|\,X_1+\cdots+X_n\equiv X} \rho[X] \quad \text{where} \quad \rho[X] = \sum_{i=1}^{n} \frac{1}{\alpha_i} \log \mathbb{E}\left[e^{\alpha_i X_i}\right]. \tag{4}$$

See Gerber (1979); this problem is studied further in Gerber & Pafumi (1998), as well as in Sections 5 and 6 of Deprez & Gerber (1985).

The optimal way to partition X over the participating companies, see Gerber (1979), Section 5.5, is to take $X_i \equiv \beta_i X / \sum \beta_j$. Here the $\beta_i := \alpha_i^{-1}$ are the *risk tolerances*. In this case, $\rho[X] = \beta \log \mathbb{E}\left[e^{X/\beta}\right]$ with $\beta := \sum \beta_j$. Using Hölder's inequality, it is easy to prove that this allocation actually produces a minimal total premium (4).

Note that because of the translation invariance of exponential premiums, introducing fixed side payments d_i, that is, replacing X_i by $X_i + d_i$ with $\sum d_i = 0$, does not influence the total premium paid. So the optimal reallocation found in Gerber (1979) is not unique.

4.2 Reinsurance principles

A particular example of risk transfer is reinsurance. The reinsurance market is a market with a restricted number of players and a large, though limited, financial capacity. The mechanisms in this market follow their own paradigms. In this section, the focus is a priori on stop-loss contracts. In Section 4 of Deprez & Gerber (1985) and in Chan & Gerber (1985), such a restriction is not made a priori.

We consider an insurer who wants to cede "the tail" of his risk and performs an analysis to find the optimal retention in a stop-loss reinsurance contract. In such a contract, the insurer transfers the risk $(X-d)_+$ to the reinsurer, while retaining the risk $X-(X-d)_+$, with d the retention (level) of the contract. Let the price of a stop-loss reinsurance contract for a given risk X and a retention d be denoted by $\rho[X; d]$. Furthermore, let π denote the risk measure of the insurer. Then, the insurer faces the following optimization problem:

$$\inf_{d} \pi\left[X - (X-d)_+ + \rho[X; d]\right].$$

Given π and ρ, the optimal retention can readily be derived.

It is well-known that stop-loss reinsurance is optimal from the viewpoint of the insurer. That is, if $r : \mathbb{R} \to \mathbb{R}_+$ denotes the payoff function of a reinsurance contract, assuming $0 \leq r(x) \leq x$ since gains on insurance claims are generally forbidden, then for any r the equality $\mathbb{E}[r(X)] = \mathbb{E}[(X-d)_+]$ implies

$$\mathbb{E}[u(r(X) - X)] \leq \mathbb{E}[u((X-d)_+ - X)],$$

with u non-decreasing and concave. This says that stop-loss reinsurance is preferred to any other form of reinsurance by all risk averse expected utility maximizers. But notice that this conclusion crucially depends on the implicit assumption that if $\mathbb{E}[r(X)] = \mathbb{E}[(X-d)_+]$ holds, that is, the reinsurance contracts under consideration have equal expected coverage, then the price charged for contract r is the same as the price $\rho[X; d]$ charged for the stop-loss contract. Clearly, this will generally not be the case in practice. To verify this, reconsider (1). Since $r(X) \leq_{\mathrm{SL}} (X-d)_+$ whenever $\mathbb{E}[r(X)] = \mathbb{E}[(X-d)_+]$, we find that for all risk averse expected utility maximizers

$$\mathbb{E}[u(w + \rho - r(X))] \geq \mathbb{E}[u(w + \rho - (X-d)_+)].$$

Hence, the equivalent utility premium for $r(X)$ is smaller than the equivalent utility premium for $(X-d)_+$.

In the framework of the previous section, the insurer might also consider a more general problem. Let the risk X be decomposed as follows:

$$X \equiv X_1 + X_2 + X_3 + X_4,$$

with

$$\begin{aligned}
X_1 &\equiv X 1_{\{X \leq 0\}} : \text{ the profit layer;} \\
X_2 &\equiv \min(X 1_{\{X > 0\}}, c) : \text{ the reinsurance layer with retention 0 and cap } c; \\
X_3 &\equiv \min((X 1_{\{X > 0\}} - c)_+, \rho[X]) : \text{ the economic capital layer;} \\
X_4 &\equiv (X - c - \rho[X])_+ : \text{ the residual risk layer.}
\end{aligned}$$

We note that the r.v.'s X_1, X_2, X_3, X_4 are *comonotonic* as they are all increasing functions of X; we defer until Section 6 a formal definition of this strong positive dependence notion. By specifying the cost of economic capital, the stop-loss reinsurance pricing principle and the risk measure of the insurer, an optimal retention policy can be derived. We note that the different layers of the risk X are measured or priced by different agents and thus by different principles.

5 A bridge between actuarial and financial pricing principles

A versatile tool to unify actuarial and financial pricing principles is the time-honored *Esscher transform*; the interested reader is referred to Bühlmann (1980), Goovaerts, De Vylder & Haezendonck (1984), Gerber & Shiu (1994, 1996), Bühlmann *et al.* (1996, 1998), Jacod & Shiryaev (2003) and Goovaerts & Laeven (2008) for various contributions in this direction.

The Esscher transform was originally introduced in Esscher (1932), who suggested to apply the well-known Edgeworth approximation to this transform rather than to the original distribution function.

It can be shown that the Esscher transform also appears in an optimal premium problem when the risk measure is exponential-like. We consider the expected utility framework and assume the utility function $u : \mathbb{R} \to \mathbb{R}$ to be exponential with coefficient of absolute risk aversion α, so that the corresponding equivalent utility premium is the exponential premium. Gerber (1974, 1985) proved within the expected utility paradigm that the equivalent utility premium is the exponential premium if and only if it is additive for independent r.v.'s; see Goovaerts *et al.* (2004b) and Goovaerts, Kaas & Laeven (2010) for more general results.

Now suppose that, rather than adopting the equivalent utility principle, we use expected utility maximization with an exponential utility function in an optimal premium problem that, for any r.v. X, only allows premiums of the form $\mathbb{E}[\varphi(X)X]/\mathbb{E}[\varphi(X)]$. Here $\varphi(\cdot)$ is a positive, continuous and strictly increasing function. Premiums of this form can also be expressed as expectations under a transformed probability measure and hence can be regarded as financial pricing principles, consistent with a no arbitrage setup.

It is proven in Goovaerts *et al.* (1984) (for a slightly improved proof see Kaas *et al.*, 2008, Theorem 5.4.3), that the optimal choice has $\varphi(x) \propto e^{\alpha x}$. This entails that the best premium under these constraints is an Esscher(α) premium. The Esscher premium also has some highly undesirable properties; see, for example, Gerber (1981) and Goovaerts *et al.* (2004b).

6 Axiomatic characterizations

It was noticed already in the introduction of this paper that an important tool to justify (or criticize) a risk measure is an axiomatic characterization. In this section we will sketch a general method to axiomatize mixtures (i.e., probability weighted averages) of distribution characterizing functions (that have a one-to-one correspondence to a distribution function). Examples of such functions are the pdf, the mgf, the cumulant generating function (cgf), as well as the stop-loss transform and the TVaR function.

An example of such a mixture is the family of *spectral risk measures*. They are defined as probability weighted averages of TVaRs. In the sequel, we write $\mathrm{TVaR}_h[X]$ also when $h = 1$, understanding the limit $\lim_{h\to 1} \mathrm{TVaR}_h[X] = \text{ess.sup}[X]$ in that case. One easily verifies that the r.v.'s X and Y are equal in distribution if, and only if, $\mathrm{TVaR}_h[X] = \mathrm{TVaR}_h[Y]$ for all $h \in [0, 1]$. Hence, the Tail-Value-at-Risk is what we call a distribution characterizing function. The interested reader is referred to Kusuoka (2001) for an axiomatic characterization of the family of spectral risk measures. Using the general method introduced below, we present a new and different axiomatic characterization of spectral risk measures.

We first introduce the notion of *comonotonicity*. We state the following (equivalent) definitions for a pair of r.v.'s to be comonotonic; we follow Denneberg (1994), Proposition 4.5.

Definition 6.1. *A pair of r.v.'s $X, Y : \Omega \to \mathbb{R}$ is said to be comonotonic if either of the following two equivalent conditions hold:*

(i) there is no pair $\omega_1, \omega_2 \in \Omega$ such that $X(\omega_1) < X(\omega_2)$ while $Y(\omega_1) > Y(\omega_2)$;

(ii) there exists a function $Z : \Omega \to \mathbb{R}$ and non-decreasing functions f, g such that

$$X(\omega) = f(Z(\omega)), \qquad Y(\omega) = g(Z(\omega)), \qquad \text{for all } \omega \in \Omega.$$

Comonotonicity is a very strong positive dependence notion. Definition 6.1 (ii) points out that if r.v.'s are comonotonic, all multivariate problems are reduced to univariate problems. The interested reader is referred to Dhaene *et al.* (2002) for an elaborate study of comonotonicity and its applications in insurance and finance.

In the remainder of this section, we fix a probability measure $\mathbb{P} \in \mathcal{P}$ on $(\Omega, \mathcal{F})$. For a given r.v. X we consider a *real-valued* distribution characterizing function $\varphi_X : (a, b) \to \mathbb{R}$. We assume that φ_X is non-decreasing and normalized, satisfying $\varphi_c(h) = c$ for all real c, $a \leq h \leq b$.

We introduce a continuous r.v. H_0 with a strictly increasing distribution function F_{H_0} supported on $[a, b]$ and having positive jumps at both a and b. Here, $[a, b]$ is taken such that it coincides with the (closed) domain of φ_X (not X). In the case where $a = -\infty$ and $b = \infty$, the r.v. H_0 is defective.

We consider the r.v. $\varphi_X(H_0)$. Since φ_X depends on the distribution of X rather than on the r.v. X itself, we can assume without loss of generality that H_0 is independent of the indices used.

Then we introduce a functional $\xi_{H_0} \equiv \xi$ that assigns a real number to any r.v. $\varphi_X(H_0)$. We state the following set of axioms that ξ must satisfy:

A1. If $\varphi_X(H_0) \leq \varphi_Y(H_0)$ a.s., then $\xi[\varphi_X(H_0)] \leq \xi[\varphi_Y(H_0)]$;

A2. $\xi[c] = c$, for all real c;

A3. $\xi[\varphi_X(H_0) + \varphi_Y(H_0)] = \xi[\varphi_X(H_0)] + \xi[\varphi_Y(H_0)]$;

A4. If $\varphi_{X_n}(H_0)$ converges a.s. to $\varphi_X(H_0)$, then $\lim_{n\to\infty} \xi[\varphi_{X_n}(H_0)] = \xi[\varphi_X(H_0)]$.

Along the lines of the proof of Theorem 3 of Goovaerts *et al.* (2004b), the following result can be proven.

Lemma 6.1. *The functional ξ satisfies the set of axioms A1–A4 if and only if there exists some non-decreasing function $G : [a,b] \to [0,1]$ such that*

$$\xi[\varphi_X(H_0)] = G(a)\varphi_X(a) + \int_{(a,b)} \varphi_X(h)\,\mathrm{d}G(h) + (1 - G(b))\varphi_X(b).$$

In Goovaerts *et al.* (2004b) it is demonstrated that a true equivalence statement formally requires an extension of the class of functions for which axioms A1–A4 should hold; see Goovaerts *et al.* (2004b) for details.

6.1 An example

As an example, we provide a new axiomatic characterization of the family of spectral risk measures. We state the following set of axioms that a risk measure π must satisfy:

B1. If $\mathrm{TVaR}_h[X] \leq \mathrm{TVaR}_h[Y]$ for all $h \in [0,1]$, then $\pi[X] \leq \pi[Y]$;

B2. $\pi[c] = c$ for all real c;

B3. $\pi[X+Y] = \pi[X] + \pi[Y]$, when X and Y are comonotonic;

B4. If X_n converges weakly to X, with $\text{ess.sup}[X_n] \to \text{ess.sup}[X]$, then $\lim_{n\to\infty} \pi[X_n] = \pi[X]$.

First, notice that axiom B1 implies law invariance. Notice furthermore that if $X \leq_{\text{st}} Y$, then $\text{TVaR}_h[X] \leq \text{TVaR}_h[Y]$ for all $0 \leq h \leq 1$. Thus, axiom B1 guarantees monotonicity of the risk measure π.

Comonotonic additivity as an axiom was imposed by Greco (1982), Schmeidler (1986, 1989) and Yaari (1987); see also Theorem 3 of Anger (1977).

Then we state the following corollary:

Corollary 6.1. *The risk measure π satisfies the set of axioms B1–B4 if and only if there exists some non-decreasing function $G : [0,1] \to [0,1]$ such that*

$$\pi[X] = G(0)\mathbb{E}[X] + \int_{(0,1)} \text{TVaR}_h[X] \, \mathrm{d}G(h) + (1 - G(1))\text{ess.sup}[X].$$

Proof: The proof of this corollary follows from Lemma 6.1 by defining $\pi[X] := \xi[\varphi_X(H_0)]$ and taking $\varphi_X(h) = \text{TVaR}_h[X]$. $\square$

6.2 Another example

Reconsider the setup of Section 4.1. Suppose that in this pool, the risk aversion of company i is $\alpha_i := h/\mathbb{E}[X_i]$, for a given number $h \geq 0$, and where we restrict ourselves to X_i such that $\mathbb{E}[X_i] > 0$. This degree of risk aversion is meaningful because the bigger $\mathbb{E}[X_i]$, the larger the company, and the lower its risk aversion: recall that in Bühlmann's (1985) allocation model the coefficient of absolute risk aversion is proportional to $\frac{1}{w}$; see Section 2 below (2). Now assuming that the mean risk $\mathbb{E}[X_i]$ contributed to the pool by company i is proportional to its initial wealth, it is reasonable that the risk aversion of company i is indeed of the form $h/\mathbb{E}[X_i]$ for some constant $h \geq 0$. For the total pool, we have a risk aversion $\alpha := h/\mathbb{E}[X]$, with $\mathbb{E}[X] > 0$. In this case, the optimal allocation is determined by the ratio of individual expected loss to aggregate expected loss, which at its turn is equal to the ratio of individual initial wealth to aggregate initial wealth.

Then each participant of the pool calculates his own risk $Y \equiv X_i$ (the same for the total pool with $Y \equiv X$) as

$$\phi_Y(h) = \frac{\mathbb{E}[Y]}{h} \log \mathbb{E}\left[e^{hY/\mathbb{E}[Y]}\right]. \tag{5}$$

Since $\phi_Y(h)/\mathbb{E}[Y]$ is the exponential premium for Y when the monetary unit equals $\mathbb{E}[Y]$, we call $\phi_Y(h)$ the *unit-mean exponential premium.*

As is well-known, the exponential premiums for $Y/\mathbb{E}[Y]$ increase with the risk aversion h, and the net premium 1 results as a limit when $h \to 0$. So we may write $\phi_Y(0) = \mathbb{E}[Y]$.

Because the mgf $m_Y(t) = \exp(t\phi_Y(t\phi_Y(0)))$ directly follows from $\phi_Y(\cdot)$, there is a one-to-one correspondence between ϕ_Y and the cdf F_Y as well.

Now we assume that h is an outcome of some r.v. H_0 satisfying the above-mentioned assumptions, with $a = 0$ and $b = \infty$. We state the following set of axioms that a risk measure π must satisfy:

C1. If $\phi_X(h) \leq \phi_Y(h)$ for all $h \geq 0$, then $\pi[X] \leq \pi[Y]$;

C2. $\pi[c] = c$, for all real c;

C3. For a r.v. Z having unit-mean exponential premiums equal to $\phi_Z(h) = \phi_X(h)+\phi_Y(h)$ for all $h \geq 0$, we have $\pi[Z] = \pi[X] + \pi[Y]$;

C4. If X_n converges weakly to X, with ess.sup$[X_n] \to$ ess.sup$[X]$, then $\lim_{n\to\infty} \pi[X_n] = \pi[X]$.

Remark 6.1. *As regards C3, we note the following. Suppose that X and Y are both compound Poisson and independent. Write $\mu_1 = \mathbb{E}[X]$, λ_1 for the Poisson parameter in X, and $m_1(\cdot)$ for the mgf of the claim severities. Analogously, index 2 pertains to Y, index 3 to Z. We want to prove that the total unit-mean exponential premiums $\phi_X(h)+\phi_Y(h)$ are those of the compound Poisson r.v. Z (having mean $\mu_3 = \phi_Z(0) = \phi_X(0)+\phi_Y(0) = \mu_1+\mu_2$) with as its mean number of claims*

$$\lambda_3 = \frac{\mu_1\lambda_1 + \mu_2\lambda_2}{\mu_1 + \mu_2},$$

and for the mgf of the severity distribution:

$$m_3(h/\mu_3) = \frac{\mu_1\lambda_1}{\mu_1\lambda_1 + \mu_2\lambda_2} m_1(h/\mu_1) + \frac{\mu_2\lambda_2}{\mu_1\lambda_1 + \mu_2\lambda_2} m_2(h/\mu_2).$$

This expresses that the severities, say $\widetilde{Z}$, in Z satisfy

$$\widetilde{Z}/\mu_3 \overset{d}{=} I\widetilde{X}/\mu_1 + (1-I)\widetilde{Y}/\mu_2,$$

where I is Bernoulli($\frac{\mu_1\lambda_1}{\mu_1\lambda_1+\mu_2\lambda_2}$), independent of $\widetilde{X}$ and $\widetilde{Y}$.

By the form of the cgf of a compound Poisson distribution and relation (5), we only have to prove that for these choices,

$$\frac{\mu_1}{h}\lambda_1[m_1(h/\mu_1) - 1] + \frac{\mu_2}{h}\lambda_2[m_2(h/\mu_2) - 1] = \frac{\mu_3}{h}\lambda_3[m_3(h/\mu_3) - 1].$$

This is easily checked.

Since the compound Poisson distributions and their limits make up the class of infinitely divisible distributions, see for example Feller (1971), this means that for all independent infinitely divisible X and Y, a r.v. Z such as in axiom C3 can indeed be found.

The class of infinitely divisible distributions is very wide. Apart from Poisson, negative binomial, normal, gamma and inverse Gaussian distributions, the n^{th} root of the mgfs of all of which produces another mgf of the same type, it also includes Student, Fisher, Pareto, Gumbel, Weibull, stable, lognormal, logistic, half-Cauchy and more distributions. See for example Sato (1999, Chapter 2, Section 8), and the references therein.

With this set of axioms for r.v.'s $X \in \mathcal{X}$, we can define a corresponding set of axioms for the corresponding r.v.'s $\phi_X(H_0)$, $X \in \mathcal{X}$, as follows:

D1. If $\phi_X(H_0) \leq \phi_Y(H_0)$ a.s., then $\xi[\phi_X(H_0)] \leq \xi[\phi_Y(H_0)]$;

D2. $\xi[\phi_c(H_0)] = \xi[c] = c$, for all real c;

D3. If $\mathbb{P}[\phi_Z(H_0) = \phi_X(H_0) + \phi_Y(H_0)] = 1$, then $\xi[\phi_Z(H_0)] = \xi[\phi_X(H_0)] + \xi[\phi_Y(H_0)]$;

D4. If $\phi_{X_n}(H_0)$ converges a.s. to $\phi_{X_n}(H_0)$, then $\lim_{n\to\infty} \xi[\phi_{X_n}(H_0)] = \xi[\phi_X(H_0)]$.

Then we state the following corollary:

Corollary 6.2. *The risk measure π satisfies the set of axioms C1–C4 if and only if there exists some non-decreasing function $G : [0,\infty] \to [0,1]$ such that*

$$\pi[X] = G(0)\mathbb{E}[X] + \int_{(0,\infty)} \phi_X(h)\,\mathrm{d}G(h) + (1 - G(\infty))\text{ess.sup}[X]. \tag{6}$$

Proof: The proof of this corollary follows from Lemma 6.1 by defining $\pi[X] := \xi[\varphi_X(H_0)]$ and taking $\varphi_X(h) = \phi_X(h)$. □

Remark 6.2. *Axiom D3 expresses comonotonic additivity for the unit-mean exponential premiums with random risk aversion. For the case of normal risks X and Y to be measured, we can translate this into a requirement differing in general from comonotonic additivity. Indeed, let (X,Y) be bivariate normal with parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho$, and let Z satisfy $\phi_Z(h) = \phi_X(h) + \phi_Y(h)$ for all h. As the normal cgfs are quadratic functions, it is easy to see that Z is also normal, with parameters satisfying*

$$\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y]; \qquad \frac{\text{Var}[Z]}{\mathbb{E}[Z]} = \frac{\text{Var}[X]}{\mathbb{E}[X]} + \frac{\text{Var}[Y]}{\mathbb{E}[Y]},$$

where we suppose $\mathbb{E}[X], \mathbb{E}[Y] > 0$*. Now let* $T = X + Y$*, then* $\mathbb{E}[T] = \mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[Z]$ *always holds, but* $\mathrm{Var}[Z] = \mathrm{Var}[T]$ *implies that*

$$\mathrm{Cov}[X,Y] = \frac{1}{2}\left[\frac{\mathbb{E}[Y]}{\mathbb{E}[X]}\mathrm{Var}[X] + \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\mathrm{Var}[Y]\right],$$

so for the correlation ρ *of* (X,Y) *we have, writing* $v[U] = \sqrt{\mathrm{Var}[U]}/\mathbb{E}[U]$ *for the coefficient of variation of r.v.* U*:*

$$\rho = \frac{1}{2}\big(v[X]/v[Y] + v[Y]/v[X]\big).$$

So for the risks X *and* Y *themselves, under Axioms D1–D4, comonotonic additivity is required for risks having the same coefficient of variation. The derivation shows that if* (X,Y) *is bivariate normal and* Z *having* $\phi_Z(h) = \phi_X(h) + \phi_Y(h)$ *has the same distribution as* $X + Y$*, then* ρ *should have the specific value in the last equality. That is, additivity is required in case the correlation* ρ *has some specific value depending on the coefficients of variation. See also Goovaerts et al. (2004a).*

Remark 6.3. *The family of risk measures characterized by (6) is subadditive and positively homogeneous.*

Proof: The latter is trivial; for the former, recall that the cgf $\kappa(h)$ is convex, because the Esscher(h) premium, which is just $\kappa'_X(h)$, increases in h. So for each $h \geq 0$ (recall that we assumed $\mathbb{E}[X] > 0$ and $\mathbb{E}[Y] > 0$),

$$\begin{aligned}
&\mathbb{E}[X+Y]\log\mathbb{E}\left[\exp h\left(\frac{X+Y}{\mathbb{E}[X+Y]}\right)\right] \\
&\quad = \mathbb{E}[X+Y]\log\mathbb{E}\left[\exp h\left(\frac{\mathbb{E}[X]}{\mathbb{E}[X+Y]}\frac{X}{\mathbb{E}[X]} + \frac{\mathbb{E}[Y]}{\mathbb{E}[X+Y]}\frac{Y}{\mathbb{E}[Y]}\right)\right] \\
&\quad \leq \mathbb{E}[X+Y]\left\{\frac{\mathbb{E}[X]}{\mathbb{E}[X+Y]}\log\mathbb{E}\left[\exp h\left(\frac{X}{\mathbb{E}[X]}\right)\right] + \frac{\mathbb{E}[Y]}{\mathbb{E}[X+Y]}\log\mathbb{E}\left[\exp h\left(\frac{Y}{\mathbb{E}[Y]}\right)\right]\right\} \\
&\quad = \mathbb{E}[X]\log\mathbb{E}\left[\exp h\left(\frac{X}{\mathbb{E}[X]}\right)\right] + \mathbb{E}[Y]\log\mathbb{E}\left[\exp h\left(\frac{Y}{\mathbb{E}[Y]}\right)\right]. \qquad \square
\end{aligned}$$

Remark 6.4. *In the context of Section 4.1, subadditivity implies that the exponential premium charged by the pool is smaller than the sum of the exponential premiums charged by the individual insurers, with equality when the total risk* X *is redistributed allocating* $(\mathbb{E}[X_i]/\mathbb{E}[X])X$ *to insurer* i*. Notice that the premium principle of the pool is now truly* *the same mapping as the premium principle of the individual insurer because*

$$\begin{aligned}\sum_{i=1}^{n} 1/\alpha_i &:= \sum_{i=1}^{n} \mathbb{E}[X_i]/h \\ &= \mathbb{E}[X]/h \\ &=: 1/\alpha,\end{aligned}$$

while in Section 4.1 this need not in general be the case.

Remark 6.5. *For small values of* h*, by Taylor's expansion, the loading contained in the premium (5) can be interpreted in terms of the coefficient of variation* $\sqrt{\mathrm{Var}[Y]}/\mathbb{E}[Y]$ *as follows:*

$$\begin{aligned}\phi_Y(h) &= \frac{\mathbb{E}[Y]}{h}\log \mathbb{E}[e^{hY/\mathbb{E}[Y]}] \\ &= \frac{\mathbb{E}[Y]}{h}\kappa_Y(h/\mathbb{E}[Y]) \\ &= \frac{\mathbb{E}[Y]}{h}\left(\frac{h}{\mathbb{E}[Y]}\mathbb{E}[Y] + \frac{1}{2}\frac{h^2}{(\mathbb{E}[Y])^2}\mathrm{Var}[Y] + \cdots\right) \\ &\approx \mathbb{E}[Y]\left(1 + \frac{1}{2}h\frac{\mathrm{Var}[Y]}{(\mathbb{E}[Y])^2}\right),\end{aligned}$$

where κ_Y *denotes the cgf of* Y*.*

7 Incomplete information

In many practical situations, no a priori probability measure will be given. When only partial (i.e., incomplete) information on the r.v. is available, for example mean and variance, one may restrict oneself to the set of "admissible" distribution functions, satisfying the constraints implied by the information available, and then maximize the risk measure over this set. Doing so, one obtains what is often called a "worst case risk measurement", which can be regarded as a prudent assessment of the risk.

To illustrate this approach, we consider again the solvency capital problem, now under incomplete information. In particular, we assume that the mean of the r.v. is available and equal to c and, in addition, we assume that the distribution is unimodal with given mode m.

For a fixed and given value of ρ, we then solve

$$V(\rho;\delta,m) = \delta\rho + \sup_{F\in\mathcal{G}_{\mathrm{u(m)}}}\left(\int_{a-}^{b}(x-\rho)_+\,\mathrm{d}F(x) \;\middle|\; \int_{a-}^{b} x\,\mathrm{d}F(x) = c,\ \int_{a-}^{b}\mathrm{d}F(x) = 1\right),$$

in which δ denotes the rate of the cost of solvency capital, $\mathcal{G}_{\mathrm{u(m)}}$ denotes the set of all unimodal distributions with fixed mode m, and a and b are the infimum and the supremum of the support of the distribution F, respectively. The value of this problem at its solution can be derived analytically (see De Vylder, 1982) and is given by

$$V(\rho;\delta,m)=\delta\rho+\left(c-\frac{1}{2}(a+m)\right)\frac{(b-\rho)^2}{(b-m)(b-a)}. \tag{7}$$

To derive the optimal amount of solvency capital, which is denoted by ρ^*, $V(\rho;\delta,m)$ must be minimized with respect to ρ. It is not difficult to verify from (7) that ρ^* is given by

$$\rho^*=b-\delta\frac{(b-m)(b-a)}{2c-(a+m)}.$$

8 Conclusion

It is generally agreed that no model is a true representation of reality. A model can only be viewed as an image of reality. The same holds for risk measures. In the financial and actuarial approach of this topic, no explicit distinction is made between measuring risk, insurance risk capital allocation problems, solvency capital principles and economic aspects in price setting for different actors. Different actors, and researchers, have different thoughts about what an appropriate system of axioms for risk measuring should be. In this paper we tried to explain the hierarchical relation between a risk measure, a decision principle and an uncertainty measure induced by incomplete information on the set of admissible distributions. All of them can be regarded as functionals assigning a real number to a random variable. Risk measures operate on the first level, decision principles operate on the second level, for example, by optimization of the total cost of an economic transaction between two or more actors in the economic reality.

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