The Concept of Comonotonicity in Actuarial Science and Finance: Applications

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Abstract

In an insurance context, one is often interested in the distribution function of a sum of random variables. Such a sum appears when considering the aggregate claims of an insurance portfolio over a certain reference period. It also appears when considering discounted payments related to a single policy or a portfolio, at different future points in time. The assumption of mutual independence between the components of the sum is very convenient from a computational point of view, but sometimes not a realistic one. In *The Concept of Comonotonicity in Actuarial Science and Finance: Theory*, we determined approximations for sums of random variables, when the distributions of the components are known, but the stochastic dependence structure between them is unknown or too cumbersome to work with. Practical applications of this theory will be considered in this paper. Both papers are to a large extent an overview of recent research results obtained by the authors, but also new theoretical and practical results are presented.

1 Introduction

In Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2001) we presented an overview of the actuarial literature on the problem how to make decisions in case we have a sum of random variables with given marginal distribution functions but of which the stochastic dependence structure is unknown or too cumbersome to work with. We proved that the convex-largest sum of the components of a random vector with given marginals is obtained in case the random vector $(X_1, X_2, \ldots, X_n)$ has the comonotonic distribution, which means that each two possible outcomes $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ of $(X_1, X_2, \ldots, X_n)$ are ordered componentwise.

In this paper, we will present several applications of the concept of comonotonicity in the field of actuarial science and finance. The notations, assumptions and results used throughout this paper are presented in the above mentioned companion paper and will not be repeated here. References to equations and theorems presented in the first paper will be denoted by adding a "T" to the relevant equation or theorem number.

As a theoretical example of the concept of comonotonicity in an insurance context, consider a portfolio of n risks $X_1, X_2, \ldots, X_n$, identically distributed, with cdf F and finite variance σ^2, say. If the risks are mutually independent, it is well-known that

$$Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{\sigma^2}{n} \to 0$$

as n goes to infinity. If the risks are comonotonic, then

$$Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = Var\left[\frac{1}{n}\sum_{i=1}^{n} F^{-1}(U)\right] = \sigma^2,$$

where U is uniformly distributed on $[0,1]$. Hence, in case of comonotonic risks, risk pooling has completely no risk reducing effect: adding an additional risk to the portfolio will not reduce the variance of the average risk.
In general, the risks of an insurance portfolio $(X_1, X_2, \ldots, X_n)$ will not exhibit the extreme comonotonic dependence structure. However, in the presence of positive dependencies between the individual risks, assuming independence might lead to an underestimation of the probability of large total claims for the portfolio. In this case the technique of risk pooling might not be as effective as expected. On the other hand, resorting to comonotonicity is a conservative approach in case the structure of dependence is unknown to the actuary.

In Section 2 we will give examples of comonotonic random variables occuring in an actuarial or financial environment. In Section 3 we give some numerical examples how to construct convex lower and upper bounds for sums of random variables. The evaluation of cash-flows in case of a lognormal discount process is considered in Section 4. In Section 5, we derive lower and upper bounds for the price of arithmetic Asian options.

2 Comonotonic random variables

In this section, we will describe several situations in an actuarial or financial context where comonotonic random variables emerge.

2.1 Options and insurance

In a financial environment, the most straigthforward examples of comonotonicity occur when considering pay-offs of derivative securities. Such pay-off functions are strongly (positively or negatively) dependent of the value of the underlying asset. This makes them useful instruments for hedging.

To be specific, let $A(t)$ be the value of a security at a future time t, $t \geq 0$. Consider a European call option on this security, with expiration date $T \geq 0$ and exercise price K. The pay-off of the call-option at time K is given by $(A(T) - K)_+$. The pay-off of the portfolio consisting of the security and the call-option is given by $\left(A(T),\ (A(T) - K)_+\right)$, which is a comonotonic random vector, since the pay-off of the option is a non-decreasing function of the value of the underlying security at the expiration date. Hence, the holder of the security who buys the call option increases his potential gains, at the cost of the option premium. One immediately finds that $A(T) + (A(T) - K)_+$ stochastically dominates A_T.

On the other hand, if the holder of the security decides to write the call option, the pay-off of his portfolio at time T is given by $\left(A(T),\ -(A(T) - K)_+\right)$. The pay-off $-(A(T) - K)_+$ is a non-increasing function of $A(T)$. One says that $\left(A(T),\ -(A(T) - K)_+\right)$ is a "counter-monotonic" random vector: if one of the components increases, then the other one decreases, see e.g. Embrechts, McNeil & Straumann (2001). Writing the call option induces an immediate gain (the option premium), at the cost of reducing the maximal gain on the underlying security. $A(T) - (A(T) - K)_+$ is stochastically dominated by $A(T)$.

A European put option with the same characteristics has a pay-off equal to $(K - A(T))_+$. In this case, the holder of the security who buys the put option has a portfolio pay-off $\left(A(T), (K - A(T))_+\right)$ which is a counter-comonotonic random vector. Buying the put-option reduces the maximal loss at the cost of the option premium. Note that $A(T)$ is stochastically dominated by $A(T) + (K - A(T))_+$.

Holding the security and writing the put option leads to a portfolio pay-off given by $\left(A(T), -(K - A(T))_+\right)$, which is a comonotonic random vector. This strategy induces an immediate gain (the option premium) but increases the potential losses on the underlying security. $A(T) - (K - A(T))_+$ is stochastically dominated by $A(T)$.

Let $(X_1, X_2, \ldots, X_n)$ be an insurance portfolio of individual risks X_i which are not assumed to be mutually independent. As mentioned in the Introduction, in the presence of positive dependencies, the risks might not be pooled as effectively as expected. The insurer could reduce the aggregate risk of his portfolio by financial hedging techniques. He could buy a financial contract with payments Y such that Y and $X_1 + X_2 + \ldots + X_n$ are comonotonic (or as comontonic as possible). Compensation will then be obtained as the pay-off of the financial contract will increase if the aggregate loss $X_1 + X_2 + \ldots + X_n$ increases. As an example in case of hurricane and earthquake insurance, the insurer could buy call options on the CAT-index, the Index of Catastrophe Losses of the Chicago Board of Trade. These options will be exercised by the insurer in case the level of the CAT-index is sufficiently high. In this case, investors take the position of the traditional reinsurer.
On the other hand, the insurer could also sell a financial contract with obligations for him that are negatively correlated with the aggregate loss $X_1 + X_2 + \ldots + X_n$. Compensation will then be obtained as his obligations related to the financial contract will decrease if the aggregate insurance loss increases. In the hurricane and earthquake example, the insurer could write put options on the CAT-index. These options generate a premium and will only be exercised if the CAT-index remains sufficiently low.

As another example, consider an insurance which protects house-owners against the depreciation of their property. Assume that the insurance payment is defined as a non-increasing function of some general real estate index. The risks of such a portfolio are comonotonic: they are all a non-increasing function of the same random variable (the real estate index). Such a portfolio cannot be considered as a traditional "insurance portfolio" where increasing the number of policies reduces the volatility of the average risk. The insurer will have to use financial hedging techniques to cope with the risk of such a portfolio. He could e.g. buy put options on the real estate index. In this case, the income of the put options and the insurance portfolio payments are comonotonic. The insurer could also write call options on the real estate index. This strategy generates an income (the option premiums) for the insurer, while the option payments and the insurance payments are counter-monotonic.

2.2 Life annuities - deterministic discount process

Consider a *life annuity* a_x on a life (x) which pays an amount of 1 at the end of each year, provided the insured is still alive at that time. Let T be a non-negative continuous random variable representing the remaining lifetime of (x). The distribution function of T is denoted by $F_T(t) = {}_tq_x$, $(t \geq 0)$. Further, the ultimate age of the life table is denoted by ω, this means that $\omega - x$ is the first remaining lifetime of (x) for which ${}_{\omega-x}q_x = 1$, or equivalently, $F_T^{-1}(1) = \omega - x$. Assume that discounting is performed with a deterministic interest rate r. The present value at policy issue of the future payments is denoted by S and equals the sum of the present values of the payments in the respective years:

$$S = \sum_{i=1}^{\lceil \omega - x \rceil - 1} X_i \tag{1}$$

where $\lceil . \rceil$ is the ceiling function, i.e. $\lceil x \rceil$ is the smallest integer greater than or equal to x, and where the random variables X_i are given by

$$X_i = v^i \; I\left(T > i\right), \tag{2}$$

with $v = (1 + r)^{-1}$ and $I(\cdot)$ denoting the indicator function, i.e. $I(c) = 1$ if the condition c is true and $I(c) = 0$ if it is not.

All X_i are non-decreasing functions of the remaining life time T, which means that the payment vector $\underline{X}$ is comonotonic. For any $0 < p < 1$, we find from Theorem T.1 that $F_{X_i}^{-1}(p) = v^i \; I\left(F_T^{-1}(p) > i\right) = v^i \; I\left(i \leq \lceil F_T^{-1}(p) \rceil - 1\right)$. Hence, letting $\sum_{i=a}^{b} x_i = 0$ if $a > b$, we find from Theorem T.6:

$$F_S^{-1}(p) = \sum_{i=1}^{\lceil \omega - x \rceil - 1} F_{X_i}^{-1}(p) = \sum_{i=1}^{\lceil F_T^{-1}(p) \rceil - 1} v^i, \qquad 0 < p < 1, \tag{3}$$

It is straightforward to verify that this expression also holds for $p = 1$.

An expression for the inverse distribution function of S can also be derived in another way, since S can be written as

$$S = \sum_{i=1}^{\lceil T \rceil - 1} v^i. \tag{4}$$

The function g defined by

$$g(y) = \sum_{i=1}^{\lceil y \rceil - 1} v^i$$

for all non-negative values of y, is non-decreasing and left-continuous. Application of Theorem T.1 leads to

$$F_S^{-1}(p) = F_{g(T)}^{-1}(p) = g\left[F_T^{-1}(p)\right], \qquad 0 < p < 1.$$

Hence, for any $0 < p < 1$, we find (3).
An expression for the cdf of S follows from (T.45):

$$F_S(x) = \sup\left\{ p \in (0,1) \mid \sum_{i=1}^{\lceil F_T^{-1}(p) \rceil - 1} v^i \leq x \right\}. \tag{5}$$

2.3 Risk sharing schemes

Let X be a non-negative random variable denoting the risk a person faces during the insurance period. An insurance contract is an agreement between this person (the insured) and the insurer where the insurer promises to pay an amount $\varphi(X)$ in case the claim amount equals X, where φ is a non-negative function, defined for all possible outcomes of X. Then $X - \varphi(X)$ is the part of the claim retained by the insured. It is reasonable to require that $\varphi(x)$ and $x - \varphi(x)$ are non-decreasing functions on the set of all possible outcomes of X. This is equivalent to requiring that both risk sharing partners have to bear more (or at least as much) if the actual claim x increases. If the benefit function φ is differentiable, both requirements reduce to the condition $0 \leq \varphi'(x) \leq 1$ for all possible outcomes x of X.

From characterization (T.23) for comonotonicity one finds that if both partners of the risk sharing scheme $(\varphi(X), X - \varphi(X))$ have to bear more if the claim amount increases, then the random vector $(\varphi(X), X - \varphi(X))$ is comonotonic.
Also the opposite can be proven: if the risk sharing scheme $(\varphi(X), X - \varphi(X))$ is comonotonic, then both partners have to bear more if the claim amount increases, except perhaps on a set with zero probability. Indeed, the comonotonicity of $(\varphi(X), X - \varphi(X))$ implies that there exists a support A of X such that the set $\{(\varphi(x), x - \varphi(x)) \mid x \in A\}$ is comonotonic. This implies that the

functions $\varphi(x)$ and $x - \varphi(x)$ must be simultaneously non-decreasing or non-increasing on A. Since the sum of the functions $\varphi(x)$ and $x - \varphi(x)$ equals the non-decreasing function x, we must have that the functions $\varphi(x)$ and $x - \varphi(x)$ are both non-decreasing on A. This proves the stated result.

As a first example of a risk sharing scheme, consider a *deductible coverage* (or *stop-loss coverage*) where the benefit function is defined by:

$$\varphi(x) = (x - d)_+ \text{ for some } d \geq 0. \tag{6}$$

It is straightforward to verify that $(\varphi(X), X - \varphi(X)) = ((X - d)_+, \min(X, d))$ which is a comonotonic random vector.
In case of *coinsurance* (or *quota share coverage*), the benefit function is defined by

$$\varphi(x) = \alpha\ x, \qquad \alpha \in [0, 1]\,. \tag{7}$$

Since both $\alpha\, x$ and $(1-\alpha)x$ increase with x, the random vector $(\alpha\ X, (1 - \alpha)\ X)$ is comonotonic.
A *coverage with a maximal limit* is defined by

$$\varphi(x) = \min\{x, d\}\,, \qquad d \geq 0. \tag{8}$$

In this case, $(\varphi(X), X - \varphi(X)) = \big(\min\{X, d\}\,,\ (X - d)_+\big)$ which is comonotonic.
Also coverages combining the three forms above, such as

$$\varphi(x) = \min\{\alpha\ (x - d_1)_+,\ d_2\}\,, \qquad \alpha \in [0, 1]\,, d_1, d_2 \geq 0 \tag{9}$$

can be seen to lead to a comonotonic risk sharing scheme.
An example of a risk sharing scheme which does not lead to comonotonic risks is a policy with a *franchise deductible* where the benefit function is defined by

$$\varphi(x) = x\ I(x \geq d), \qquad d \geq 0. \tag{10}$$

We find that $(\varphi(X), X - \varphi(X)) = (X\ I(X \geq d),\ X\ I(X < d))$ which is in general not a comonotonic random vector.

All the examples above describe risk sharing schemes between insurer and insured, but they can also be interpreted as risk sharing schemes between insurer and reinsurer.
An example of a reinsurance scheme which does not necessarily lead to comonotonic risks is the *largest claim reinsurance*. Indeed, let the insurance portfolio consist of n individual risks with claim amounts $Y_1 \leq Y_2 \leq$

$\cdots \leq Y_n$ respectively (the Y_i are the order statistics corresponding to the risks in the portfolio). The risk taken by the reinsurer equals Y_n, while the risk kept by the ceding insurer is $Y_1 + Y_2 + \cdots + Y_{n-1}$. It is clear that $(Y_n,\ Y_1 + Y_2 + \cdots + Y_{n-1})$ will in general not be comonotonic.

3 Convex bounds for sums of rv's

In this section, we will illustrate the technique of deriving convex lower and upper bounds for sums of random variables, as explained in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2001), by some numerical examples. Especially, we will consider sums of normal or lognormal random variables.

Recall that a random vector $(Y_1, Y_2, \ldots, Y_n)$ has the multivariate normal distribution if and only if every linear combination of its variates has a univariate normal distribution. Now assume that $(Y_1, Y_2, \ldots, Y_n)$ has a multivariate normal distribution. Let Y and Λ be linear combinations of the variates: $Y = \sum_{i=1}^{n} \alpha_i Y_i$ and $\Lambda = \sum_{i=1}^{n} \beta_i Y_i$. Then also $(Y,\ \Lambda)$ has a bivariate normal distribution.

Further, if $(Y,\ \Lambda)$ has a bivariate normal distribution, then, conditionally given $\Lambda = \lambda$, Y has a univariate normal distribution with mean and variance given by

$$E\left[Y \mid \Lambda = \lambda\right] = E\left[Y\right] + r\left(Y, \Lambda\right) \frac{\sigma_Y}{\sigma_\Lambda} \left(\lambda - E\left[\Lambda\right]\right) \tag{11}$$

and

$$Var\left[Y \mid \Lambda = \lambda\right] = \sigma_Y^2\ \left(1 - r\left(Y, \Lambda\right)^2\right), \tag{12}$$

where $r\left(Y, \Lambda\right)$ is Pearson's correlation coefficient for the couple (Y, Λ).

Example 1 ***(sums of normal rv's)***

Let Y_1, Y_2 be mutually independent $N(0,1)$ random variables. Obviously, $S = Y_1 + Y_2$ is $N(0,2)$. For the convex order bounds for S, we will consider conditioning random variables of the type $\Lambda = Y_1 + aY_2$ for some real a. The conditional distribution of Y_1, given $\Lambda = \lambda$, is $N\left(\frac{\lambda}{1+a^2}, \frac{a^2}{1+a^2}\right)$.

This means that for the conditional expectation $E[Y_1|\Lambda]$ and for the random variable $F^{-1}_{Y_1|\Lambda}(U)$, with U uniform(0,1) and independent of Λ, we get

$$E[Y_1|\Lambda] = \frac{\Lambda}{1+a^2} \text{ and } F^{-1}_{Y_1|\Lambda}(U) = E[Y_1|\Lambda] + \frac{|a|\Phi^{-1}(U)}{\sqrt{1+a^2}}.$$

In line with $E[Y_1 + aY_2 \mid \Lambda] \equiv \Lambda$, we also get

$$E[Y_2 \mid \Lambda] = \frac{a\Lambda}{1+a^2} \text{ and } F^{-1}_{Y_2|\Lambda}(U) = E[Y_2|\Lambda] + \frac{\Phi^{-1}(U)}{\sqrt{1+a^2}}.$$

Both $F^{-1}_{Y_1|\Lambda}(U)$ and $F^{-1}_{Y_2|\Lambda}(U)$ have $N(0,1)$ distributions. Their U-dependent parts are comonotonous. For the comonotonic upper bound S^c, the improved upper bound S^u and the lower bound S^l as derived in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2001), we get

$$\begin{aligned}
S = Y_1 + Y_2 &\sim N(0,2), \\
S^l = E[Y_1 + Y_2 \mid \Lambda] = \frac{1+a}{1+a^2}\Lambda &\sim N\left(0, \frac{(1+a)^2}{1+a^2}\right), \\
S^u = \frac{1+a}{1+a^2}\Lambda + \frac{1+|a|}{\sqrt{1+a^2}}\Phi^{-1}(U) &\sim N\left(0, \frac{(1+a)^2 + (1+|a|)^2}{1+a^2}\right), \\
S^c \stackrel{d}{=} 2Y_1 &\sim N(0,4).
\end{aligned}$$

For some special choices of a, we get the following distributions for the lower and upper bounds S^l and S^u:

$$\begin{aligned}
a = 0 &: N(0,1) \leq_{cx} S \leq_{cx} N(0,2), \\
a = 1 &: N(0,2) \leq_{cx} S \leq_{cx} N(0,4), \\
a = -1 &: N(0,0) \leq_{cx} S \leq_{cx} N(0,2), \\
|a| \to \infty &: N(0,1) \leq_{cx} S \leq_{cx} N(0,2).
\end{aligned}$$

Note that the actual distribution of S is $N(0,2)$, so the best convex lower bound ($a = 1$) and the best upper bound ($a \leq 0$ or $a \to \infty$) coincide with S. Of course taking $|a| \to \infty$ gives the same results as taking $\Lambda = Y_2$. The variance of S^l can be seen to have a maximum at $a = +1$, a minimum at $a = -1$. On the other hand, $Var[S^u]$ also has a maximum at $a = 1$, and minima at $a \leq 0$ and $a \to \infty$. So the best lower bound in this case is attained for $\Lambda = S$, the worst for Λ and S independent. The best improved upper bound is found by taking $\Lambda = Y_1$, $\Lambda = Y_2$, or any $a < 0$, including the case $a = -1$ with Λ and S independent; the worst, however, by taking $\Lambda = S$.

To compare the variance of the stochastic upper bound S^u with the variance of S boils down to comparing $cov\left(F^{-1}_{Y_1|\Lambda}(U), F^{-1}_{Y_2|\Lambda}(U)\right)$ with $cov\,(Y_1, Y_2)$. It is clear that, in general, the optimal choice for the conditioning random variable Λ will depend on the correlation of Y_1 and Y_2. If this correlation equals 1, any Λ results in $S \overset{d}{=} S^u \overset{d}{=} S^c$. In our case where Y_1 and Y_2 are mutually independent, the optimal choice proves to be taking $\Lambda \equiv Y_1$, which corresponds to $a = 0$ or $\Lambda \equiv Y_2$, which corresponds to $a \to \infty$, thus ensuring that S and S^u coincide. But also any $a < 0$ leads to $S \overset{d}{=} S^u$.▼

Example 2 ***(sums of lognormal rv's)***

As a second example, consider a simple special case of the theory dealt with in the next section. We present it here for the reader's convenience, just as an illustration. Take Y_1 and Y_2 independent $N(0,1)$ random variables. Look at the sum of $X_1 = e^{Y_1+Y_2} \sim$ lognormal$(0,2)$, and $X_2 = e^{Y_2} \sim$ lognormal$(0,1)$. $S = X_1 + X_2$ can then be interpreted as the value at time 2 of investing a unit amount at time 0 and one at time 1, where the investment returns in year 1 and 2 are given by Y_1 and Y_2 respectively. For the lower bound S^l, take $\Lambda = Y_1 + Y_2$. Note that $E[X_1|\Lambda] = e^{\Lambda}$, while $Y_2|\Lambda = \lambda \sim N(\frac{1}{2}\lambda, \frac{1}{2})$, hence

$$E[e^{Y_2}|Y_1 + Y_2 = \lambda] = m(1; \frac{1}{2}\lambda, \frac{1}{2}),$$

where $m(t; \mu, \sigma^2) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ is the $N(\mu, \sigma^2)$ moment generating function. This leads to

$$E[e^{Y_2}|\Lambda] = e^{\frac{1}{2}\Lambda + \frac{1}{4}}.$$

So the lower bound is

$$S^l = E[X_1 + X_2|\Lambda] = e^{\Lambda} + e^{\frac{1}{2}\Lambda + \frac{1}{4}}.$$

The upper bound S^c follows from $(X_1^c, X_2^c) \overset{d}{=} (e^{\sqrt{2}Z}, e^Z)$ for $Z \sim N(0,1)$. The improved upper bound S^u has as a first term again e^{Λ}, and as second term $e^{\frac{1}{2}\Lambda + \frac{1}{2}\sqrt{2}Z}$, with Z and Λ mutually independent. All terms occurring in the bounds given above are lognormal random variables defined in terms of

Λ and Z, which are mutually independent, so the variances of the bounds are easy to compute. We have, as the reader may verify,

$$\begin{aligned}
(E[S])^2 &= e + 2e^{\frac{3}{2}} + e^2, \\
E[(S^l)^2] &= e^{\frac{3}{2}} + 2e^{\frac{5}{2}} + e^4, \\
E[S^2] = E[(S^u)^2] &= e^2 + 2e^{\frac{5}{2}} + e^4, \\
E[(S^c)^2] &= e^2 + 2e^{\frac{3}{2}+\sqrt{2}} + e^4.
\end{aligned}$$

Hence,

$$\begin{aligned}
Var[S^l] &= 64.374, \\
Var[S] = Var[S^u] &= 67.281, \\
Var[S^c] &= 79.785.
\end{aligned}$$

So a close stochastic lower bound S^l for S is obtained by conditioning on $Y_1 + Y_2$. The improved upper bound S^u for this case proves to be very good. Indeed, as $S \leq_{cx} S^u$ while the variances are equal, the improved upper bound S^u has the same distribution as S. This result could be expected because by conditioning on $\Lambda = \lambda$, the random variable $X_1 = e^{\Lambda}$ is fixed and hence the random vector (X_1, X_2) is comonotonic. Recall that the lower bound will be the best if the conditioning random variable Λ resembles S as closely as possible, see Section T.5.3. Approximating e^{Y_2} and $e^{Y_1+Y_2}$ by $1 + Y_2$ and $1 + Y_1 + Y_2$ respectively, we see that $S \approx 2 + Y_1 + 2Y_2$, hence we could expect that taking $Y_1 + 2Y_2$ instead of $Y_1 + Y_2$ as our conditioning random variable might lead to a better lower bound. This is not true however since the variance of the lower bound is 61.440 in this case. It proves that the optimal lower bound obtained by conditioning on random variables of type $Y_1 + aY_2$ is reached for $a = 1.27$ and the variance of S^l is then 66.082.▼

Example 3 ***(sums of conditionally independent rv's)***

Consider a home fire insurance portfolio consisting of n risks X_i with $\Pr[X_i = 0] = 0.90$, $\Pr[X_i = 1] = 0.04$ and $\Pr[X_i = 2] = 0.06$. Assume that the claim amounts depend on the weather conditions during the insurance year. Let Λ be a Bernoulli random variable which equals 1 (with probability $\frac{1}{3}$) in case of a dry hot summer. Assume that we know the conditional

distributions, given Λ, of the risks X_i : Let $\Pr[X_i = 0, 1 \mid \Lambda = 0] = 0.94,\ 0.06$ and $\Pr[X_i = 0,\ 2 \mid \Lambda = 1] = 0.82,\ 0.18$. Hence, a dry hot summer leads to higher claim frequencies and severities. Further, we assume that, given $\Lambda = \lambda$, the risks X_i are conditionally independent.
We find that $Var\,[\Lambda] = \frac{2}{9}$ and $Var\,[X_i] = 0.2544$. The distribution of the comonotonic upper bound S^c follows from

$$S^c \stackrel{d}{=} \sum_{i=1}^{n} F_{X_i}^{-1}(U) \stackrel{d}{=} n\ X_1,$$

from which we find that $Var\,[S^c] = 0.2544\ n^2$.
The cdf of the lower bound S^l follows from

$$S^l \stackrel{d}{=} n\ E\,(X_1 \mid \Lambda) \stackrel{d}{=} n\ \ \{0.06\,(1-\Lambda) + 0.36\ \Lambda\}\,.$$

Hence, $Var\left[S^l\right] = 0.02\ n^2$. Note that under the assumption that the X_i are mutually independent (which is clearly unrealistic in this case) , we find that the variance of $S^\perp = X_1^\perp + \cdots + X_n^\perp$ is given by $0.2544\ n$ which will be smaller than the variance of the lower bound if $n > 12$.
The cdf of the improved upper bound S^u follows from

$$S^u \stackrel{d}{=} n\ F_{X_1|\Lambda}^{-1}(U) \stackrel{d}{=} n\ X_1,$$

so that we find that $S^u \stackrel{d}{=} S^c$ in this case.
In order to compute the exact variance of $S = X_1 + \ldots + X_n$, note that

$$Var\,[S] = Var\left[S^l\right] + E\,[Var\,[S \mid \Lambda]]\,.$$

We find that $Var\,[X_i \mid \Lambda] = \{0.0564(1-\Lambda) + 0.5904\Lambda\}$. We assumed that conditionally, given $\Lambda = \lambda$, the random variables X_i are mutually independent. So we find $E\,[Var\,[S \mid \Lambda]] = n\ E\,[Var\,[X_i \mid \Lambda]] = 0.2344\ n$. We finally get that $Var\,[S] = (0.02n + 0.2344)\ n$. Hence,

$$\frac{Var(S)}{Var(S^l)} = 1 + \frac{11.72}{n}$$

indicating that the performance of the lower bound improves as the the size of the portfolio increases. Even for a relatively small portfolio, the lower bound seems to perform very well.▼

The results of the previous example can be generalized. Indeed, consider a portfolio of n risks X_i. Assume that for any possible outcome λ of Λ we have that, conditionally given $\Lambda = \lambda$, the risks X_i are identically distributed, but not necessarily mutually independent.
In this case, we find that $\sum_{i=1}^n F_{X_i}^{-1}(U) \stackrel{d}{=} \sum_{i=1}^n F_{X_i|\Lambda}^{-1}(U) \stackrel{d}{=} nX_1$, where U and Λ are mutually independent and U is uniform(0,1) distributed. Hence,

$$S^c \stackrel{d}{=} S^u \stackrel{d}{=} nX_1, \qquad Var\left[S^c\right] = Var\left[S^u\right] = n^2 \; Var\left[X_1\right]. \tag{13}$$

For the lower bound S^l, we find

$$S^l \stackrel{d}{=} n \; E\left[X_1 \mid \Lambda\right], \qquad Var\left[S^l\right] = n^2 \; Var\left[E\left[X_1 \mid \Lambda\right]\right]. \tag{14}$$

Let us now assume that conditionally, given $\Lambda = \lambda$, the risks X_i are iid. An expression for the variance of the sum $S = X_1 + \ldots + X_n$ is obtained as follows:

$$\begin{aligned} Var\left[S\right] &= Var\left[S^l\right] + E\left[Var\left[S \mid \Lambda\right]\right] \\ &= n^2 \; Var\left[E\left[X_1 \mid \Lambda\right]\right] + n \; E\left[Var\left[X_1 \mid \Lambda\right]\right], \end{aligned}$$

so that we find

$$\frac{Var\left[S^c\right]}{Var\left[S\right]} = \frac{1 + \frac{E[Var[X_1|\Lambda]]}{Var[E[X_1|\Lambda]]}}{1 + \frac{1}{n}\frac{E[Var[X_1|\Lambda]]}{Var[E[X_1|\Lambda]]}} \tag{15}$$

which is an increasing function of the volume of the portfolio, with limiting value $1 + \frac{E[Var[X_1|\Lambda]]}{Var[E[X_1|\Lambda]]}$. This means that the larger the portfolio, the worse the relative performance of the comonotonic (and the improved) upper bound.
For the lower bound however, we find that

$$\frac{Var\left[S\right]}{Var\left[S^l\right]} = 1 + \frac{1}{n}\frac{E\left[Var\left[X_1 \mid \Lambda\right]\right]}{Var\left[E\left[X_1 \mid \Lambda\right]\right]}. \tag{16}$$

Hence, the more the variance of the individual risks X_i is caused by $Var\left[E\left[X_1 \mid \Lambda\right]\right]$, the better the lower bound will perform. For a sufficiently large portfolio, the lower bound S^l will perform very well.

Defining z as the ratio $wn/(wn+v)$ where $w = Var[E[X_1 \mid \Lambda]]$ and $v = E[Var[X_1 \mid \Lambda]]$ denote the *between-variance* and the *within-variance* respectively, we can rewrite (15) and (16) as

$$Var[S^c] = (z + (1-z)n)\, Var[S] \quad \text{and} \quad Var[S^l] = z\, Var[S].$$

The factor $z \in [0, 1]$ can be interpreted as a measure for the goodness-of-fit when S is replaced by $E[S \mid \Lambda]$: The larger z, the better the lower bound performs. Maximum performance (i.e. $z = 1$) is achieved if $n \to \infty$, $w = Var[X_1]$ or $v = 0$. For a portfolio of a given size n, we also have that the larger z, the better the comonotonic upper bound will perform.

4 Provisions for future payment obligations - Lognormal discount process

4.1 Approximate evaluation of provisions

Consider a series of deterministic payments $\alpha_1, \alpha_2, \cdots, \alpha_n$, of arbitrary sign, that are due at times $1, 2, \cdots, n$ respectively. We want to find an answer to the following question: "What is the amount of money required at time 0 in order to be able to meet these future obligations $(\alpha_1, \alpha_2, \cdots, \alpha_n)$?" We will call this amount the provision, or depending on the situation at hand, the (prospective) reserve or the required capital. Of course, the level of the provision will strongly depend on the way how this amount will be invested. Let us assume that the provision will be invested such that it generates a stochastic return Y_j in year j, $j = 1, 2, \ldots, n$, i.e. an amount of 1 at time $j-1$ will grow to e^{Y_j} at time j. The discount factor over the period $[0, i]$ is then given by $e^{-(Y_1+Y_2+\cdots+Y_i)}$, because this stochastic amount will exactly grow to an amount 1 at time i. The distribution function of the random variable

$$S = \sum_{i=1}^{n} \alpha_i \; e^{-(Y_1+Y_2+\cdots+Y_i)}. \tag{17}$$

will help us to determine the provision W_0. E.g. we could determine this provision as $W_0 = F_S^{-1}(0.99)$, such that there is a 99% probability that we can meet our future obligations, which means that there is a 99% probability that after the last payment at time n, we will have a non-negative amount left.

In this section, we will assume that the return vector $(Y_1, Y_2, \cdots, Y_n)$ has a multivariate normal distribution. The random variable S is then a linear combination of dependent lognormal random variables. In any realistic return model, it is impossible to determine the distribution function of S analytically. Therefore, we will derive the convex upper and lower bounds

S^c, S^u and S^l of S. By defining random variables X_i and $Y(i)$ by

$$\begin{aligned} Y(i) &= Y_1 + Y_2 + \cdots + Y_i, &(18)\\ X_i &= e^{-Y(i)}, &(19) \end{aligned}$$

the stochastic provision S can be written as $S = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n$. Note that if all α_i are positive, then the support of S is situated in the region $\left[F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right] = (0, +\infty)$, if all α_i are negative, then $\left[F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right] = (-\infty, 0)$, and if the α_i have mixed signs, then $\left[F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right] = (-\infty, +\infty)$. In the following theorem, we derive approximations for (the distribution function of) the stochastic provision S, as explained in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2001).

Theorem 1 *Let S be given by (17), where the random vector $(Y_1, Y_2, \cdots, Y_n)$ has a multivariate normal distribution. Consider the conditioning random variable $\Lambda = \sum_{i=1}^n \beta_i \; Y_i$. Then the lower bound S^l, the improved upper bound S^u and the comonotonic upper bound S^c are given by*

$$\begin{aligned} S^l &= \sum_{i=1}^n \alpha_i \; e^{-E[Y(i)] - r_i \; \sigma_{Y(i)} \; \Phi^{-1}(V) + \frac{1}{2}\left(1-r_i^2\right)\sigma_{Y(i)}^2}, &(20)\\ S^u &= \sum_{i=1}^n \alpha_i \; e^{-E[Y(i)] - r_i \; \sigma_{Y(i)} \; \Phi^{-1}(V) + sign(\alpha_i) \; \sqrt{1-r_i^2}\sigma_{Y(i)} \; \Phi^{-1}(U)}, &(21)\\ S^c &= \sum_{i=1}^n \alpha_i \; e^{-E[Y(i)] + sign(\alpha_i) \; \sigma_{Y(i)} \; \Phi^{-1}(U)}, &(22) \end{aligned}$$

where U and V are mutually independent uniform(0,1) random variables, Φ is the cdf of the $N(0,1)$ distribution and r_i is defined by

$$r_i = r\left(Y(i), \Lambda\right) = \frac{cov\left[Y(i), \Lambda\right]}{\sigma_{Y(i)} \; \sigma_\Lambda}. \tag{23}$$

Proof. (a) From (11) and (12), we find that conditionally, given $\Lambda = \lambda$, the random variable $-Y(i)$ is normally distributed with parameters $\mu_i = -E\left[Y(i)\right] - r_i \; \frac{\sigma_{Y(i)}}{\sigma_\Lambda} \; (\lambda - E\left[\Lambda\right])$ and $\sigma_i^2 = \left(1 - r_i^2\right)\sigma_{Y(i)}^2$. Hence, conditionally, given $\Lambda = \lambda$, the random variable X_i is lognormally distributed with parameters μ_i and σ_i^2. As $E\left[X_i \mid \Lambda = \lambda\right] = e^{\mu_i + \frac{1}{2}\sigma_i^2}$, we find

$$E\left[\alpha_i X_i \mid \Lambda\right] = \alpha_i \; e^{-E[Y(i)] - r_i \; \sigma_{Y(i)} \; \Phi^{-1}(V) + \frac{1}{2}\left(1-r_i^2\right)\sigma_{Y(i)}^2},$$

where the random variable $V \equiv \Phi\left(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right)$ is uniform(0,1). Hence, $S^l = \sum_{i=1}^n E\left[\alpha_i X_i \mid \Lambda\right]$ is given by (20).

(b) From (T.75), we find that $F^{-1}_{\alpha_i X_i|\Lambda=\lambda}(p) = \alpha_i \ e^{\mu_i + sign(\alpha_i) \ \sigma_i \ \Phi^{-1}(p)}$, with μ_i and σ_i as defined in (a). This implies

$$F^{-1}_{\alpha_i X_i|\Lambda}(p) = \alpha_i \ e^{-E[Y(i)] - r_i \ \sigma_{Y(i)} \ \Phi^{-1}(V) + sign(\alpha_i) \ \sqrt{1-r_i^2}\sigma_{Y(i)} \ \Phi^{-1}(p)}.$$

Hence, $S^u = \sum_{i=1}^n F^{-1}_{\alpha_i X_i|\Lambda}(U)$ is given by (21).

(c) The random variable X_i is lognormally distributed with parameters $-E\left[Y(i)\right]$ and $\sigma^2_{Y(i)}$. From (T.75), we find that $F^{-1}_{\alpha_i X_i}(p) = \alpha_i \ e^{-E[Y(i)] + sign(\alpha_i) \ \sigma_{Y(i)} \ \Phi^{-1}(p)}$, from which we find the expression (22) for S^c. ■

Note that in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2001), we proved that the bounds in the theorem above are ordered in the convex order sense:

$$S^l \leq_{cx} S \leq_{cx} S^u \leq_{cx} S^c. \tag{24}$$

In order to compare the cdf of $S = \sum_{i=1}^n \alpha_i \ e^{-(Y_1+Y_2+\cdots+Y_i)}$ with the cdf's of the convex order bounds S^l, S^u and S^c, we may look at their variances. So we need the correlations between the different random variables in each sum. We find the following results for the lognormal discount process considered in this section:

$$r\left[\alpha_i \ X_i, \ \alpha_j \ X_j\right] = s_{ij} \frac{e^{cov[Y(i),Y(j)]} - 1}{\sqrt{e^{\sigma^2_{Y(i)}} - 1}\sqrt{e^{\sigma^2_{Y(j)}} - 1}}; \tag{25}$$

$$r\left[E\left(\alpha_j \ X_i \mid \Lambda\right), \ E\left(\alpha_j \ X_j \mid \Lambda\right)\right] = s_{ij} \frac{e^{r_i r_j \sigma_{Y(i)} \sigma_{Y(j)}} - 1}{\sqrt{e^{r_i^2 \ \sigma^2_{Y(i)}} - 1}\sqrt{e^{r_j^2 \ \sigma^2_{Y(j)}} - 1}};$$

$$r\left[F^{-1}_{\alpha_i \ X_i|\Lambda}(U), \ F^{-1}_{\alpha_j \ X_j|\Lambda}(U)\right] = s_{ij} \frac{e^{\left[r_i r_j + s_{ij}\sqrt{1-r_i^2}\sqrt{1-r_j^2}\right]\sigma_{Y(i)}\sigma_{Y(j)}} - 1}{\sqrt{e^{\sigma^2_{Y(i)}} - 1}\sqrt{e^{\sigma^2_{Y(j)}} - 1}};$$

$$r\left[F^{-1}_{\alpha_i \ X_i}(U), \ F^{-1}_{\alpha_j \ X_j}(U)\right] = s_{ij} \frac{e^{s_{ij}\sigma_{Y(i)}\sigma_{Y(j)}} - 1}{\sqrt{e^{\sigma^2_{Y(i)}} - 1}\sqrt{e^{\sigma^2_{Y(j)}} - 1}}. \tag{26}$$

where s_{ij} is used as shorthand notation for $sign(\alpha_i \alpha_j)$. From these correlations, we can for instance deduce that if all payments α_i are positive and $r\left[Y(i), Y(j)\right] = 1$ for all i and j, then $S \overset{d}{=} S^c$. In practice, the discount factors will not be perfectly correlated. But for any realistic discount process,

$r\left[Y(i), Y(j)\right] = r\left[Y_1 + \cdots + Y_i,\ Y_1 + \cdots + Y_j\right]$ will be close to 1 provided that i and j are close to each other. This gives an indication that the cdf of S^c might perform well as an approximation for the cdf of S for such processes. This is indeed the case as will be seen in the numerical illustrations in Section 4.4.

A similar reasoning leads to the conclusion that the cdf of S^c will not perform well as a convex upper bound for the cdf of S if the payments α_i have mixed signs. This phenomenon will indeed be observed in the numerical illustrations in Section 4.4.

Note that when $S = \alpha_i\, e^{-Y(i)}$, an optimal choice for the conditioning random variable Λ is given by $\Lambda = Y(i)$, as this choice implies $S^l \stackrel{d}{=} S \stackrel{d}{=} S^u \stackrel{d}{=} S^c$.

It remains to derive expressions for the cdf's of S^l, S^u and S^c.

4.2 The cdf and the stop-loss premiums of the bounds

The quantiles of S^c follow from Theorems T.1 and T.6:

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} \alpha_i\ e^{-E[Y(i)]+sign(\alpha_i)\ \sigma_{Y(i)}\ \Phi^{-1}(p)},\ p \in (0,1)\,. \tag{27}$$

The F_{X_i} are strictly increasing and continuous. From (T.48) we have that for $F_{S^c}^{-1+}(0) < x < F_{S^c}^{-1}(1)$, $F_{S^c}(x)$ follows implicitly from solving

$$\sum_{i=1}^{n} \alpha_i\ e^{-E[Y(i)]+sign(\alpha_i)\ \sigma_{Y(i)}\ \Phi^{-1}(F_{S^c}(x))} = x. \tag{28}$$

From (T.82), we find the following expression for the stop-loss premium at retention d with $F_{S^c}^{-1+}(0) < d < F_{S^c}^{-1}(1)$ for S^c:

$$\begin{aligned} E\left[(S^c - d)_+\right] &= \sum_{i=1}^{n} \alpha_i\ e^{-E[Y(i)]+\frac{\sigma^2_{Y(i)}}{2}}\ \Phi\left[sign(\alpha_i)\ \sigma_{Y(i)} - \Phi^{-1}(F_{S^c}(d))\right] \\ &\quad -d\ \left(1 - F_{S^c}(d)\right). \end{aligned} \tag{29}$$

Expressions for the cdf and the stop-loss premiums of S^l can be obtained by following the procedure as explained in (T.101) - (T.104). Indeed, from (20), we immediately find that the cdf of S^l can be determined from

$$F_{S^l}(x) = \int_0^1 I\left(\sum_{i=1}^{n} \alpha_i\ e^{-E[Y(i)]-r_i\ \sigma_{Y(i)}\ \Phi^{-1}(v)+\frac{1}{2}\left(1-r_i^2\right)\sigma^2_{Y(i)}} \le x\right) dv,$$

while the stop-loss premiums follow from

$$E\left[\left(S^l-d\right)_+\right]=\int_0^1\left(\sum_{i=1}^n \alpha_i\ e^{-E[Y(i)]-r_i\ \sigma_{Y(i)}\ \Phi^{-1}(v)+\frac{1}{2}\left(1-r_i^2\right)\sigma_{Y(i)}^2}-d\right)_+ dv.$$

Let us now consider the special yet important case that all $\alpha_i \geq 0$ and all $r_i \geq 0$. These conditions ensure that S^l is the sum of n comonotonous random variables. Taking into account that $\Lambda = \sum_{i=1}^n \beta_i\ Y_i$ is normally distributed, we find that

$$F_\Lambda^{-1}(1-p)=E\left[\Lambda\right]-\sigma_\Lambda\ \Phi^{-1}(p),$$

and hence, from (T.97) or also from Theorems T.1 and T.6,

$$F_{S^l}^{-1}(p)=\sum_{i=1}^n \alpha_i\ e^{-E[Y(i)]+r_i\ \sigma_{Y(i)}\ \Phi^{-1}(p)+\frac{1}{2}\left(1-r_i^2\right)\ \sigma_{Y(i)}^2},\quad p\in(0,1)\,. \tag{30}$$

From (T.99), we find that for any $0<x<\infty$, $F_{S^l}(x)$ can be obtained from

$$\sum_{i=1}^n \alpha_i\ e^{-E[Y(i)]+r_i\ \sigma_{Y(i)}\ \Phi^{-1}(F_{S^l}(x))+\frac{1}{2}\left(1-r_i^2\right)\ \sigma_{Y(i)}^2}=x. \tag{31}$$

From (20) we see that S^l is the comonotonic sum of n random variables $\alpha_i\ Z_i$ where the Z_i are lognormal distributed. Hence, from (T.82), we find the following explicit expression for the stop-loss premium at retention $d>0$:

$$E\left[\left(S^l-d\right)_+\right]=\sum_{i=1}^n \alpha_i\ e^{-E[Y(i)]+\frac{1}{2}\sigma_{Y(i)}^2}\Phi\left[r_i\ \sigma_{Y(i)}-\Phi^{-1}\left(F_{S^l}(d)\right)\right]-d\ \left(1-F_{S^l}(d)\right). \tag{32}$$

Finally, we determine the cdf of S^u. Since $F_{S^u}(x \mid V=v)$ is the cdf of a sum of n comonotonic random variables, we have

$$F_{S^u|V=v}^{-1}(p)=\sum_{i=1}^n \alpha_i\ e^{-E[Y(i)]-r_i\ \sigma_{Y(i)}\ \Phi^{-1}(v)+sign(\alpha_i)\ \sqrt{1-r_i^2}\sigma_{Y(i)}\ \Phi^{-1}(p)}. \tag{33}$$

For $F_{S^u|V=v}^{-1+}(0)<x<F_{S^u|V=v}^{-1}(1)$, the conditional probabilities $F_{S^u}(x \mid V=v)$ also follow implicitly from

$$\sum_{i=1}^n \alpha_i\ e^{-E[Y(i)]-r_i\ \sigma_{Y(i)}\ \Phi^{-1}(v)+sign(\alpha_i)\ \sqrt{1-r_i^2}\sigma_{Y(i)}\ \Phi^{-1}(F_{S^u}(x|V=v))}=x. \tag{34}$$

The cdf of S^u then follows from

$$F_{S^u}(x) = \int_0^1 F_{S^u}(x \mid V = v) \, dv. \tag{35}$$

4.3 Continuous annuities

Many of the results derived for the discrete case (sums of random variables) have a continuous counterpart (integrals of random variables). Consider e.g. the continuous temporary annuity S defined by

$$S = \int_0^t \alpha(\tau) \exp\left[-\delta\tau - \sigma \; B(\tau)\right] \, d\tau \tag{36}$$

where $\{B(\tau), \tau \geq 0\}$ represents a standard Brownian motion, i.e. the process has independent and stationary increments, $B(0) = 0$ and for any $\tau \geq 0$, the random variable $B(\tau)$ is normally distributed with mean 0 and variance τ. Further, the drift δ and the volatility σ are non-negative real numbers. The payments are described by $\alpha(\tau)$ which is a non-negative and continuous function of τ.

Let $Y(\tau) = \delta\tau + \sigma B(\tau)$ and $X(\tau) = \exp\{-Y(\tau)\}$. It can be proven that $S \leq_{cx} S^c$, where the random variable S^c is defined by

$$S^c = \int_0^t F^{-1}_{\alpha(\tau)X(\tau)}(U) \, d\tau = \int_0^t \alpha(\tau) \exp\left[-\delta\tau + \sigma \, \sqrt{\tau} \; \Phi^{-1}(U)\right] \, d\tau, \tag{37}$$

where, as usual, U is a uniform(0,1) random variable.
The quantiles of S^c follow from

$$F^{-1}_{S^c}(p) = \int_0^t \alpha(\tau) \exp\left[-\delta\tau + \sigma \, \sqrt{\tau} \; \Phi^{-1}(p)\right] \, d\tau, \qquad (0 < p < 1), \tag{38}$$

which is a continuous counterpart of (T.39). The stop-loss premiums with retentions $d > 0$ follow from

$$E[(S^c - d)_+] = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma^2\tau/2} \Phi\left[\sigma\sqrt{\tau} - \Phi^{-1}(F_{S^c}(d))\right] d\tau - d(1 - F_{S^c}(d)), \tag{39}$$

where $F_{S^c}(d)$ can be obtained by solving $F^{-1}_{S^c}(F_{S^c}(d)) = d$, using (38). See (T.82) for a discrete counterpart of this expression.

In the remainder of this subsection, we consider a constant annuity. Hence, we assume that $\alpha(\tau) \equiv 1$. In order to derive a lower bound in convex order for S, we consider the conditioning random variable $\Lambda = \int_0^t e^{-\delta\tau} B(\tau) d\tau$ which is a linear transformation of a first order approximation of S. We have that Λ is normally distributed with mean 0 and variance

$$\begin{aligned}\sigma_\Lambda^2 = Var[\Lambda] &= \int_0^t \int_0^t e^{-\delta(\tau+\nu)} \min(\tau,\nu) d\tau d\nu \\ &= \frac{1}{2\delta^3} + \frac{3+2\delta t - 4e^{\delta t}}{2\delta^3 e^{2\delta t}}. \end{aligned} \tag{40}$$

Since $B(\tau)$ is a Brownian motion process, the random variable $Y(\tau) \mid \Lambda = \lambda$ is normally distributed with mean

$$E[Y(\tau)|\Lambda = \lambda] = \delta\tau + r(\tau)\sigma\sqrt{\tau}\frac{\lambda}{\sigma_\Lambda} \tag{41}$$

and variance

$$Var[Y(\tau)|\Lambda = \lambda] = \sigma^2\tau(1 - r^2(\tau)) \tag{42}$$

with $r(\tau)$ defined by

$$r(\tau) = \frac{cov[Y(\tau), \Lambda]}{\sigma_\Lambda \sigma\sqrt{\tau}} = \frac{1}{\sigma_\Lambda\sqrt{\tau}}\left[\frac{1-e^{-\delta\tau}}{\delta^2} - \frac{\tau e^{-\delta t}}{\delta}\right], \quad \tau \le t. \tag{43}$$

Analoguously to (20), it can be shown that $S^l \le_{cx} S$ where S^l is defined by

$$S^l = E[S \mid \Lambda] = \int_0^t \exp\left\{-\delta\tau - r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(V) + \frac{1}{2}\sigma^2\tau(1-r^2(\tau))\right\} d\tau \tag{44}$$

with $V = \Phi\left(\frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right)$ standard uniformly distributed.

The function $f(\tau) = cov[Y(\tau), \Lambda]$ turns out to be a non-negative function, i.e. $f(\tau) \ge 0$ for $0 \le \tau \le t$, since $f(\tau)$ is continuous and

$$\begin{aligned} f(0) &= 0 \\ f'(\tau) &= \frac{\sigma}{\delta}\left(e^{-\delta\tau} - e^{-\delta t}\right) > 0, \quad \tau < t. \end{aligned}$$

Consequently, $r(\tau)$ too is a non-negative function and the integrand in (44) is a decreasing function of V. This implies that S^l is an integral of comonotonous random variables. Hence, the quantiles of S^l follow from

$$F_{S^l}^{-1}(p) = \int_0^t \exp\left\{-\delta\tau + r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(p) + \frac{1}{2}\sigma^2\tau(1-r^2(\tau))\right\} d\tau, \tag{45}$$

for $0 < p < 1$. The stop-loss premiums of S^l with retentions $d > 0$ follow from

$$E[(S^l - d)_+] = \int_0^t e^{-\delta\tau+\sigma^2\tau/2}\Phi\left[r(\tau)\sigma\sqrt{\tau} - \Phi^{-1}(F_{S^l}(d))\right] d\tau - d(1 - F_{S^l}(d)), \tag{46}$$

where $F_{S^l}(d)$ can be obtained by solving $F_{S^l}^{-1}(F_{S^l}(d)) = d$. Similar results can be obtained in case $\alpha(\tau)$ is a more general function.

4.4 Numerical illustrations

4.4.1 Discrete annuities

In this section, we will numerically illustrate the bounds we derived for $S = \sum_{i=1}^{20} \alpha_i \; e^{-(Y_1+Y_2+\cdots+Y_i)}$. We will assume that the random variables Y_i are i.i.d. and $N(\mu, \sigma^2)$. The conditioning random variable Λ is defined as before:

$$\Lambda = \sum_{i=1}^{20} \beta_i \; Y_i, \tag{47}$$

In this case, we find

$$\begin{aligned} E\left[Y(i)\right] &= i \; \mu, & (48)\\ Var\left[Y(i)\right] &= i \; \sigma^2, & (49)\\ Var\left[\Lambda\right] &= \sigma^2 \sum_{k=1}^{20} \beta_k^2, & (50)\\ r_i &= \frac{cov\left[Y(i), \Lambda\right]}{\sigma_{Y(i)} \; \sigma_\Lambda} = \frac{\sum_{k=1}^{i} \beta_k}{\sqrt{i \sum_{k=1}^{20} \beta_k^2}}. & (51) \end{aligned}$$

In our numerical illustrations, we will choose the parameters of the normal distribution involved as follows:

$$\mu = 0.07 \qquad \sigma = 0.1$$

We will compute the lower and upper bounds for the following choice of the parameters β_i:

$$\beta_i = \sum_{j=i}^{20} \alpha_j e^{-j\mu}, \quad i = 1, \cdots, 20.$$

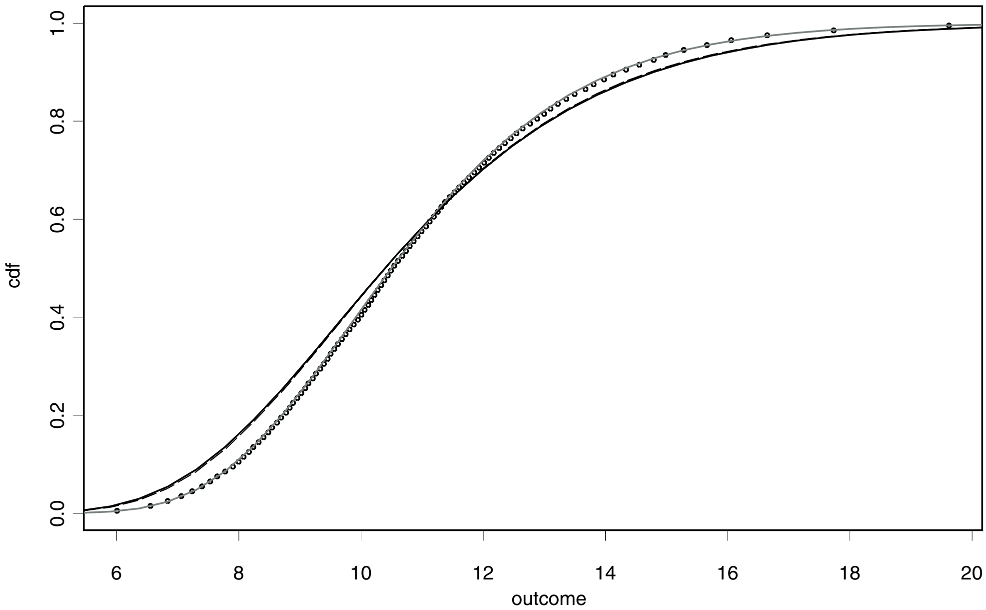

Figure 1: The cdf's of S (dotted line), S^l (solid grey line), S^u (dashed line) and S^c (solid black line); positive payments.

This choice makes Λ a linear transformation of a first order approximation to S. This can be seen from the following computation:

$$\begin{aligned} S &= \sum_{j=1}^{20} \alpha_j e^{-j\mu - \sum_{i=1}^{j}(Y_i-\mu)} \approx \sum_{j=1}^{20} \alpha_j e^{-j\mu}[1 - \sum_{i=1}^{j}(Y_i - \mu)] \\ &= C - \sum_{j=1}^{20} \alpha_j e^{-j\mu} \sum_{i=1}^{j} Y_i = C - \sum_{i=1}^{20} Y_i \sum_{j=i}^{20} \alpha_j e^{-j\mu}, \end{aligned}$$

where C is the appropriate constant. By the remarks in section 4.1, S^l will then be "close" to S, provided $(Y_i - \mu)$ is sufficiently small, or equivalently, σ is sufficiently small.

Figure 1 shows the cdf's of S, S^l, S^u and S^c for the following payments:

$$\alpha_k = 1, \quad k = 1, \cdots, 20.$$

Since $S^l \leq_{cx} S \leq_{cx} S^u \leq_{cx} S^c$, and the same ordering holds for the tails of their respective distribution functions which can be observed to cross only

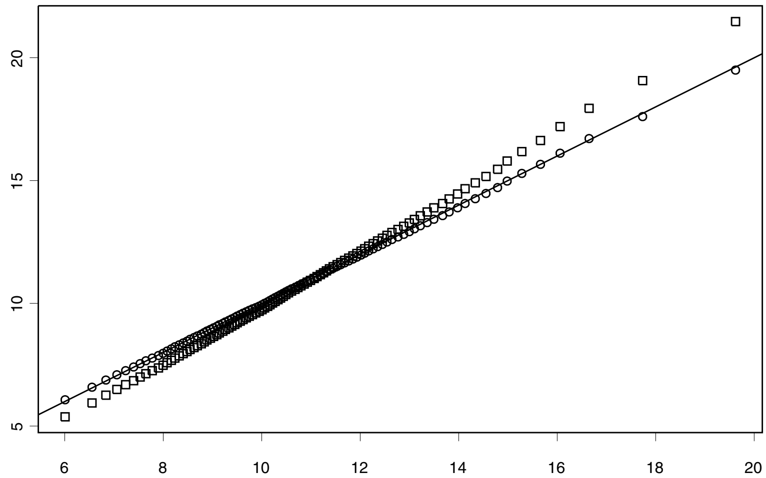

Figure 2: QQ-plot of the quantiles of S^l ($\circ$) and S^c ($\square$) versus those of S; positive payments.

p	$F_{S^l}^{-1}(p)$	'$F_S^{-1}(p)$'	$F_{S^c}^{-1}(p)$
0.95	15.4656	15.3868	16.3915
0.975	16.7108	16.7233	17.9432
0.99	18.3080	18.3942	19.9578
0.995	19.4966	19.9644	21.4739
0.999	22.2381	22.2271	25.0210

Table 1: Quantiles of S^l and S^c versus those of S; positive payments.

once, we can identify the cdf's. The dotted line is the "exact" cdf of S, which was obtained by generating 10000 quasi-random paths. We see that the cdf of S^l is very close to the distribution of S, which was to be expected because Λ is constructed such that it is "close" to S. Note that in this case S^l is a sum of comonotonic random variables, so its quantiles can be computed easily. The cdf of S^c also performs rather well as an approximation to the cdf of S. This can partially be explained by the fact that the dependency structure of the vector $(X_1, \ldots, X_n)$ is locally quasi comonotonic. Indeed, if i is close to j, then $r\left(Y(i), Y(j)\right) = \frac{\min(i,j)}{\sqrt{ij}}$ is rather close to 1. Hence, from (25) and (26), we find that $r\left[X_i,\ X_j\right]$ is close to $r\left[F_{X_i}^{-1}(U),\ F_{X_j}^{-1}(U)\right]$ if i is close to j.

We find that the improved upper bound S^u is very close to the comono-

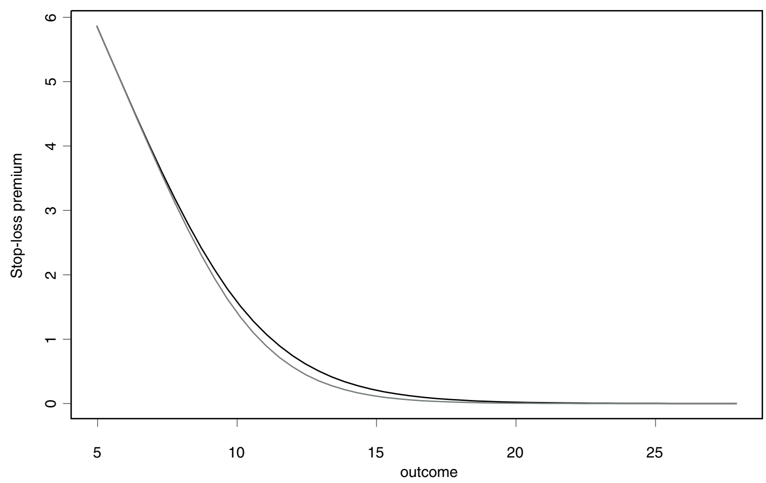

Figure 3: Stop-loss premiums for S^l (solid grey line), S (dotted line) and S^c (solid black line); positive payments.

d	$E[(S^l-d)_+]$	'$E[(S-d)_+]$'	$E[(S^c-d)_+]$
0	10.8320	10.8346	10.8320
5	5.8321	5.8346	5.8327
10	1.4136	1.4141	1.5804
15	0.1148	0.1156	0.2067
20	0.0064	0.0042	0.0216
25	0.0004	0.0003	0.0023

Table 2: Stop-loss premiums for S^l, S and S^c; positive payments.

tonic upper bound S^c. This could be expected because r_i is close to r_j for any pair (i,j) with i and j sufficiently close. This implies that for any such pair (i,j) we have that $cov\left(F^{-1}_{X_i|\Lambda}(U), F^{-1}_{X_j|\Lambda}(U)\right)$ is close to $cov\left(F^{-1}_{X_i}(U), F^{-1}_{X_j}(U)\right)$.

In order to have a better view on the behavior of the comonotonic upper bound S^c (and of the lower bound S^l) in the tails, we consider a QQ-plot where the quantiles of S^c (and of S^l) are plotted against the quantiles of S obtained by simulation. The comonotonic upper bound S^c (and the lower bound S^l) will be a good approximation for S if the plotted points $\left(F_S^{-1}(p),\ F_{S^c}^{-1}(p)\right)$ (and also $\left(F_S^{-1}(p),\ F_{S^l}^{-1}(p)\right)$) for all values of p in $(0,1)$ do not deviate too much from the straight line $y=x$. From the QQ-plot in Figure 2, we can conclude that the comonotonic upper bound (slightly) overestimates the tails

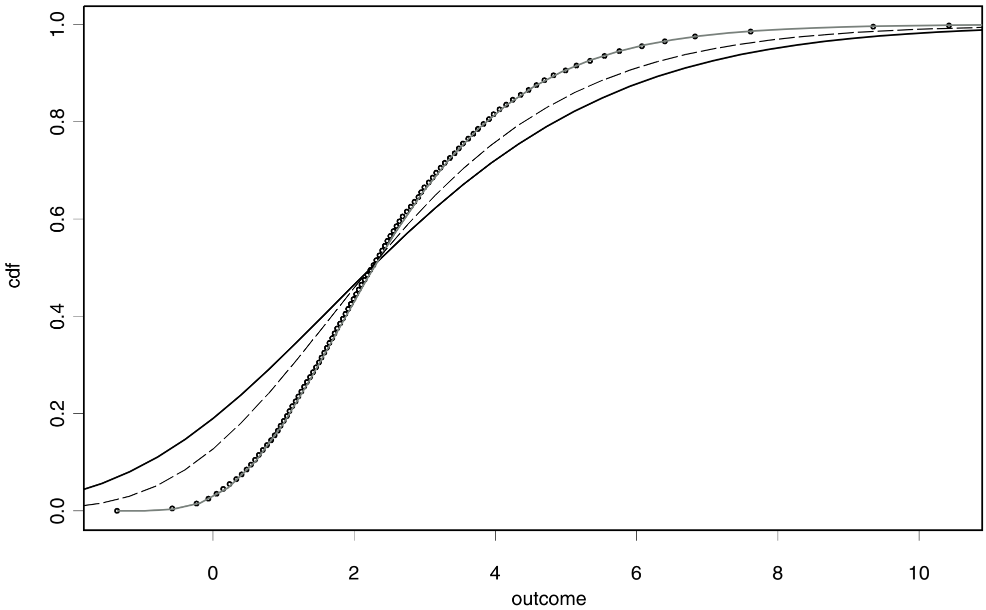

Figure 4: The cdf's of S (dotted line), S^l (solid grey line), S^u (dashed line) and S^c (solid black line); positive and negative payments.

of S, whereas the accuracy of the lower bound is extremely high. Table 1 confirms these observations.

Stop-loss premiums for S^l and S^c are compared in Figure 3. The upper and lower bound can be seen to be very close. Table 2 shows the stop-loss premiums for some retentions.

Notice that some of the simulated stop-loss premiums are not in the theoretical range $[E[(S^l - d)_+], E[(S^c - d)_+]]$. This not only demonstrates the difficulty of estimating stop-loss premiums by simulation, but it also indicates the accuracy of the bounds.

Next, we consider a series of negative and positive payments. Figure 4 shows the cdf's of S, S^l, S^u and S^c for the following payments:

$$\alpha_k = \begin{cases} -1, & k = 1, \dots, 5, \\ 1 & k = 6, \dots, 20. \end{cases} .$$

Note that the lower bound S^l is not a comonotonic sum in this case. We see that the lower bound S^l still performs very well as its cdf is almost indistinguishable from the cdf of S obtained by simulation. The comonotonic upper bound S^c performs very badly in this case. This could be expected

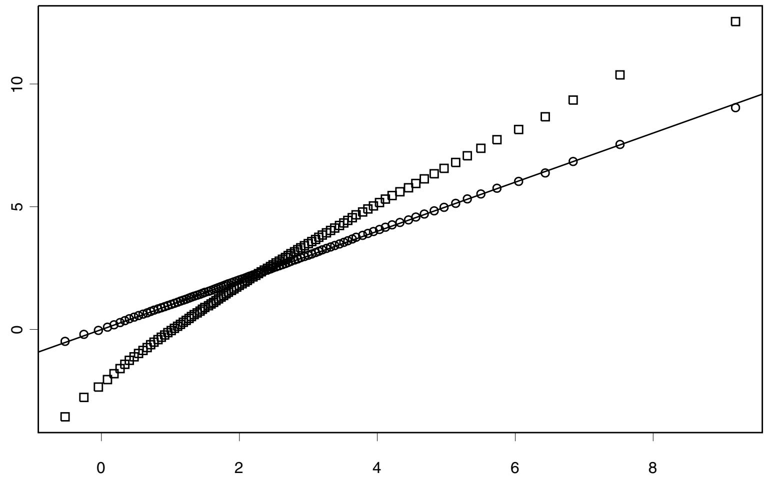

Figure 5: QQ-plot of the quantiles of S^l (○) and S^c (□) versus those of S; positive and negative payments.

p	$F_{S^l}^{-1}(p)$	'$F_S^{-1}(p)$'	$F_{S^c}^{-1}(p)$
0.95	5.8849	5.8805	7.9282
0.975	6.8400	6.8391	9.3450
0.99	8.0881	8.0206	11.1716
0.995	9.0321	9.1935	12.53998
0.999	11.2519	11.3833	15.7310

Table 3: Quantiles of S^l and S^c versus those of S; positive and negative payments.

because couples such as $(-X_5, X_6)$ are quasi counter-monotonic. Moreover, couples $(-X_k, X_l)$ with $k \leq 5$ and $l \geq 6$ exhibit a kind of negative dependency structure. The improved upper bound performs better for this cash-flow with mixed signs. These observations are confirmed by the QQ-plots in Figure 5 and Table 3.

In Figure 6, we consider the same series of payments as in Figure 4. We consider the cdf of the lower and the improved upper bound for a different choice of the conditioning random variable Λ. We choose Λ as a linear transformation of a first order approximation of $-\sum_{j=1}^{5} e^{-Y(j)}$ which is the sum

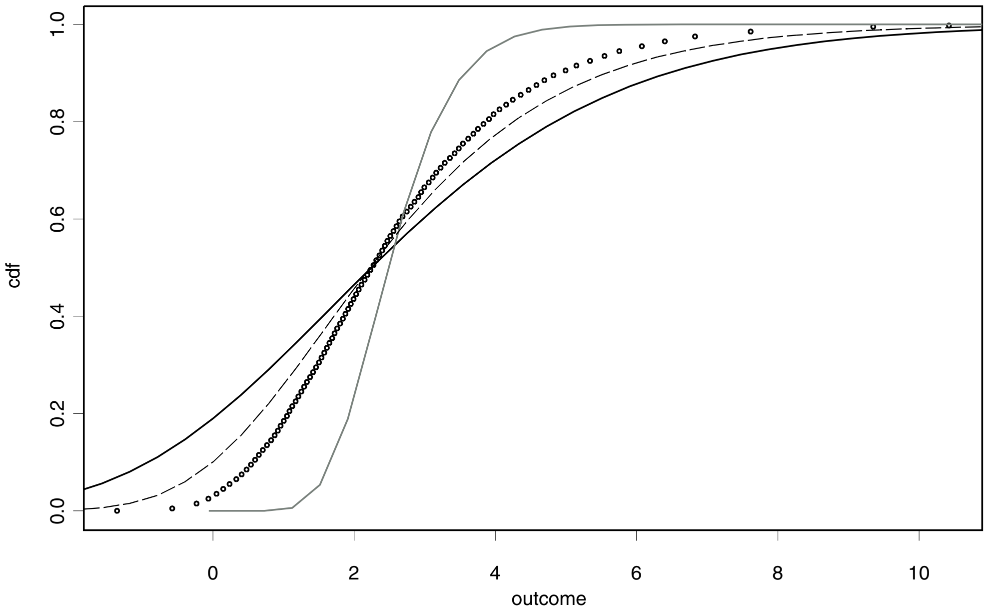

Figure 6: The cdf's of S (dotted line), S^l (solid grey line), S^u (dashed line) and S^c (solid black line); positive and negative payments, other choice of Λ.

of the negative terms in S. Hence, $\Lambda \equiv \sum_{j=1}^{5} e^{-j\mu}\, Y(j)$, or

$$\beta_i = \sum_{j=i}^{5} e^{-j\mu}, \quad i = 1, \ldots, 5$$

and $\beta_i = 0$ otherwise. The (simulated) cdf of S is the dotted line. The convex largest cdf is the comonotonic upper bound. Note that the lower bound performs worse in this case, which was to be expected because the "new" Λ is more different from S than the original one. The improved upper bound S^u performs much better in this case. This can partially be explained by the fact that conditionally on $X_1 + \ldots + X_5 = \lambda$, we have that $S = -X_1 - \ldots - X_5 + X_6 + \ldots + X_{20}$ can be approximated by a comonotonic sum. In this respect one can expect that (a first order approximation of) the sum of the negative terms will be a good choice for the conditioning variable Λ of the improved upper bound.

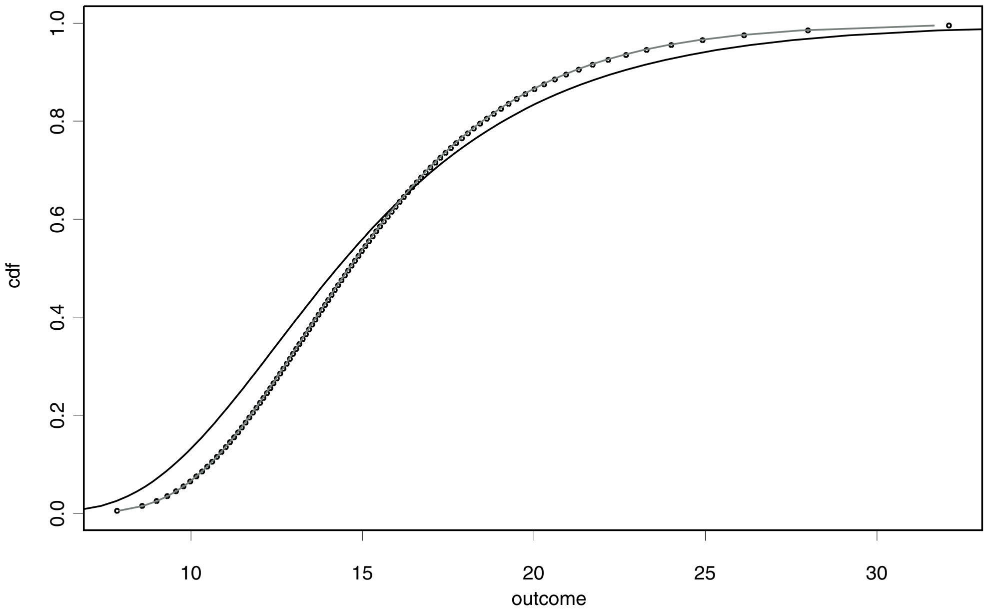

Figure 7: The cdf's of S_∞ (dotted line), S^l_∞ (solid grey line) and S^c_∞ (solid black line).

4.4.2 Continuous annuities

Consider the continuous (temporary) annuity with constant payment stream

$$S_t = \int_0^t \exp\left[-\delta\tau - \sigma B(\tau)\right]\, d\tau$$

where $B(\tau)$ represents a standard Brownian motion. For this annuity, analytic results for the distribution function are known, see e.g. De Schepper, Teunen & Goovaerts (1994). In case the time horizon t reaches infinity, the distribution function of the perpetuity S_∞ can be calculated very easily since one can prove that S_∞^{-1} is Gamma distributed with parameters $\frac{2\delta}{\sigma^2}$ and $\frac{\sigma^2}{2}$. This result can be found in Dufresne (1990) and Milevsky (1997). The probability density function of the Gamma distribution with parameters $a > 0$ and $b > 0$ is defined by

$$g(x; a, b) = \frac{1}{b\Gamma(a)} \exp\left\{\frac{-x}{b}\right\} \left(\frac{x}{b}\right)^{a-1}, \quad x > 0.$$

Hence we can compare the distribution functions of the lower bound S^l_∞ and the upper bound S^c_∞, as defined in section 4.3, with the exact distribution

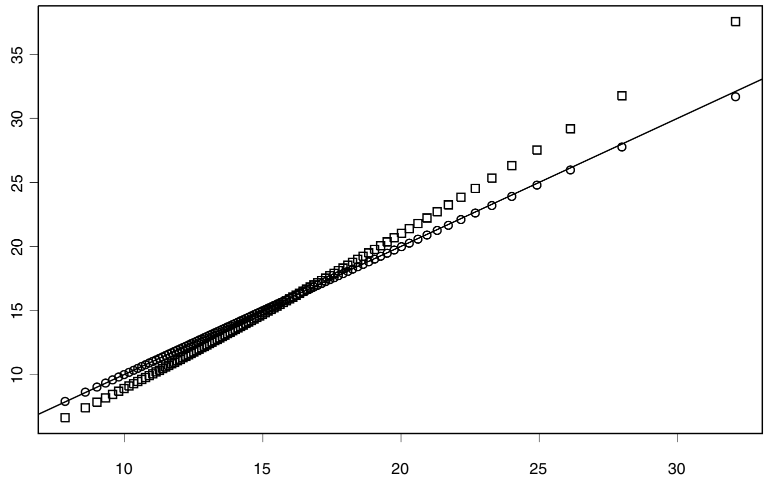

Figure 8: QQ-plot of the quantiles of S^l_∞ ($\circ$) and S^c_∞ ($\square$) versus those of S_∞.

p	$F^{-1}_{S^l_\infty}(p)$	$F^{-1}_{S_\infty}(p)$	$F^{-1}_{S^c_\infty}(p)$
0.95	23.5271	23.6297	25.7881
0.975	25.9633	26.1304	29.1857
0.99	29.1972	29.4883	33.8523
0.995	31.6810	32.0993	37.5561
0.999	37.6492	38.4953	46.8616

Table 4: Quantiles of S^l_∞ and S^c_∞ versus those of S_∞.

function of S_∞. From (40) with $t \to \infty$, it follows that the variance of the conditioning variable $\Lambda = \int_0^\infty e^{-\delta\tau} B(\tau) d\tau$ simplifies to

$$Var[\Lambda] = \frac{1}{2\delta^3}$$

while, by (43), the correlation between $Y(\tau)$ and Λ boils down to

$$r(\tau) = \frac{1}{\sigma_\Lambda \sqrt{\tau}} \frac{1 - e^{-\delta\tau}}{\delta^2}.$$

Figure 7 shows the distribution functions of S^l_∞, S^c_∞ and S_∞ for $\delta = 0.07$ and $\sigma = 0.1$. Again, the lower bound proves to be a very good approximation for the cdf of S_∞. To assess the accuracy of the bounds in the tails, we plot their

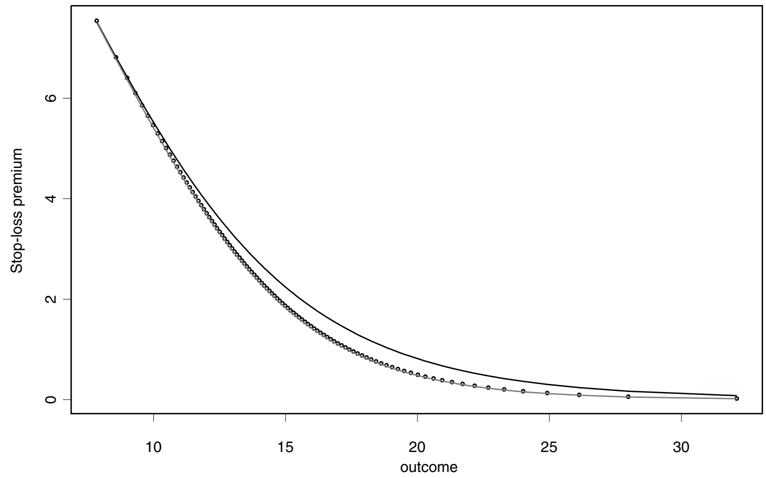

Figure 9: Stop-loss premiums for S^l_∞ (solid grey line), S_∞ (dotted line) and S^c_∞ (solid black line).

d	$E[(S^l_\infty - d)_+]$	$E[(S_\infty - d)_+]$	$E[(S^c_\infty - d)_+]$
10	5.4099	5.4457	5.5210
15	1.8354	1.8626	2.2405
20	0.4787	0.4961	0.8152
25	0.1172	0.1270	0.2973
30	0.0294	0.0342	0.1134

Table 5: Stop-loss premiums of S^l_∞, S^c_∞ and S_∞.

quantiles against those of S_∞ in Figure 8. The largest quantile ($p = 0.995$) of S^l_∞ in this QQ-plot underestimates the exact quantile by only 1.3%. Table 4 shows the numerical values for some high quantiles.

The stop-loss premiums for different choices of d are shown in Figure 9 and in Table 5.

5 Asian Options

5.1 Definitions and some theoretical results

Assume that we are currently at time 0. Consider a risky asset (a non-dividend paying stock) with prices described by the stochastic process $\{A(t),\ t \geq 0\}$, and a risk-free continuously compounded rate δ that is constant through time. In this section all probabilities and expectations have to be considered as conditional on the information available at time 0, i.e. the prices of the risky asset up to time 0. Note that in general, the conditional expectation (with respect to the physical probability measure) of $e^{-\delta t}A(t)$, given the information available at time 0, will differ from the current price $A(0)$. However, we will assume that there exists a unique "equivalent probability measure Q" such that the discounted price proces $\left\{e^{-\delta t}\ A(t),\ t \geq 0\right\}$ is a martingale under this equivalent probability measure. This implies that for any $t \geq 0$, the conditional expectation (with respect to the equivalent martingale measure) of $e^{-\delta t}A(t)$, given the information available at time 0, will be equal to the current price $A(0)$. Denoting this conditional expectation under the equivalent martingale measure by $E^Q\left[e^{-\delta t}A(t)\right]$, we have that

$$E^Q\left[e^{-\delta t}A(t)\right] = A(0), \qquad t \geq 0. \tag{52}$$

The notation $F_{A(t)}(x)$ will be used for the conditional probability that $A(t)$ is smaller than or equal to x, under the equivalent martingale measure Q, and given the information available at time 0. Its inverse will be denoted by $F_{A(t)}^{-1}(p)$.

The existence of an equivalent martingale measure is related to the absence of arbitrage in the securities market, while uniqueness of the equivalent martingale measure is related to market completeness. Two models where there exists such a unique equivalent martingale measure are the binomial three model of Cox, Ross & Rubinstein (1979) and the geometric Brownian motion model of Black & Scholes (1973).

The existence of the equivalent martingale measure allows one to reduce the pricing of options on the risky asset to calculating expected values of the discounted pay-offs, not with respect to the physical probability measure, but with respect to the equivalent martingale measure, see e.g. Harrison and Kreps (1979) or Harrison and Pliska (1981). A reference in the actuarial literature is Gerber & Shiu (1996).

A European call option on the risky asset, with exercise price K and exercise date T generates a pay-off $(A(T) - K)_+$ at time T, that is, if the price of the risky asset at time T exceeds the exercise price, the pay-off equals the difference; if not, the pay-off is zero. Note the similarity between such a pay-off and the payment on a stop-loss reinsurance contract. At current time $t = 0$ this call option will trade against a price given by

$$EC(K,T) = e^{-\delta T} E^Q \left[(A(T) - K)_+ \right] \tag{53}$$

A European-style arithmetic Asian call option with exercise date T , n averaging dates and exercise price K generates a pay-off $\left(\frac{1}{n}\sum_{i=0}^{n-1} A(T-i) - K\right)_+$ at T, that is, if the average of the prices of the risky asset at the latest n dates before T is more than K, the pay-off equals the difference; if not, the pay-off is zero. Such options protect the holder against manipulations of the asset price near the expiration date. The price of the Asian option at current time $t = 0$ is given by

$$AC(n,K,T) = e^{-\delta T} E^Q \left[\left(\frac{1}{n} \sum_{i=0}^{n-1} A(T-i) - K \right)_+ \right]. \tag{54}$$

Determining the price of an Asian option is not a trivial task, because in general we do not have an explicit analytical expression for the distribution of the average $\sum_{i=0}^{n-1} A(T-i)$. One can use Monte-Carlo simulation techniques to obtain a numerical estimate of the price, see Kemna and Vorst (1990) and F.J. Vázquez-Abad & Dufresne (1998), or one can numerically solve a parabolic partial differential equation, see Rogers & Shi (1995). But as both approaches are rather time consuming, it would be helpful to have an accurate and easily computable bound of this price. In Jacques (1996) an approximation is obtained by replacing the distribution of the sum $\sum_{i=0}^{n-1} A(T-i)$ by a more tractable one.

From the expression for the price of an arithmetic Asian call option, we see that the problem of pricing such options turns out to be equivalent to calculating stop-loss premiums of a sum of dependent random variables. This means that we can apply our previous results on bounds for stop-loss premiums in order to find accurate lower and upper bounds for the price of Asian options. Note that the lower bound that we will obtain is closely related to the lower bound derived by Rogers & Shi (1995).

5.2 Asian options, the general case

Assume that at the current time 0, the averaging has not yet started. In this case the n variables $A(T-n+1), \cdots, A(T)$ are random. Upper bounds for $AC(n,K,T)$ can be constructed as follows for any retentions K_i and K, with $K = \sum_{i=1}^{n} K_i$:

$$\begin{aligned} AC(n,K,T) &= \frac{e^{-\delta T}}{n} E^Q\left[\left(\sum_{i=0}^{n-1} A(T-i) - nK\right)_+\right] \\ &\leq \frac{e^{-\delta T}}{n} \sum_{i=0}^{n-1} E^Q\left[(A(T-i) - nK_i)_+\right] \\ &= \frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i}\; EC\left(nK_i, T-i\right). \end{aligned} \tag{55}$$

The procedure above enables us to construct an unlimited number of upper bounds for the price of an arithmetic Asian call option as an average of the prices of underlying European call options. The theory of comonotonic risks will allow us to find the best, i.e. the smallest, upper bound constructed in this way. Note that all results can be easily extended to general averaging dates $T-t_1, \ldots, T-t_n$. In this paper, however, we only consider equally spaced averaging dates in order not to complicate notation.

By introducing the notation $S^c = \sum_{i=0}^{n-1} F^{-1}_{A(T-i)}(U)$, where U is a random variable which is uniformly distributed on the unit interval, we find from Theorem T.8 that for all exercise prices K we have

$$AC(n,K,T) \leq \frac{e^{-\delta T}}{n} E^Q\left[(S^c - nK)_+\right] \tag{56}$$

From Theorem T.7 we then find that for all K with $F^{-1+}_{S^c}(0) < nK < F^{-1}_{S^c}(1)$,

$$\begin{aligned} \frac{e^{-\delta T}}{n} E^Q\left[(S^c - nK)_+\right] &= \frac{e^{-\delta T}}{n} \sum_{i=0}^{n-1} E^Q\left[\left(A(T-i) - F^{-1(\alpha)}_{A(T-i)}\left(F_{S^c}(nK)\right)\right)_+\right] \\ &= \frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i}\; EC\left(F^{-1(\alpha)}_{A(T-i)}\left(F_{S^c}(nK)\right), T-i\right) \end{aligned} \tag{57}$$

where α is determined by

$$F^{-1(\alpha)}_{S^c}(F_{S^c}(nK)) = nK. \tag{58}$$

Hence, an upper bound for the price of the Asian option $AC(n,K,T)$ with $F_{S^c}^{-1+}(0) < nK < F_{S^c}^{-1}(1)$ is given by

$$AC(n,K,T) \leq \frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i}\ EC\left(F_{A(T-i)}^{-1(\alpha)}\left(F_{S^c}(nK)\right), T-i\right) \tag{59}$$

We also find that for any retentions K_i and K with $K = \sum_{i=1}^{n} K_i$,

$$\frac{e^{-\delta T}}{n} E^Q\left[(S^c - nK)_+\right] \leq \frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i}\ EC\left(nK_i, T-i\right). \tag{60}$$

From (56), (57), (59) and (60) we find that the comonotonic dependency structure leads to the optimal upper bound for the price of an arithmetic Asian option which is weighted average of the prices of the underlying European call options as in (55).
Note that if $nK \leq F_{S^c}^{-1+}(0)$ or $nK \geq F_{S^c}^{-1}(1)$, the price of the Asian option can be determined exactly, see (T.52) and (T.53). In the first case, it is certain that the option will be in the money at the expiration date, while in the second case the option will be certainly out of the money, and hence worthless.
The upper bound in (59) can be written in terms of the usual inverses $F_{A(T-i)}^{-1}$. Indeed, one can prove that

$$\begin{aligned} AC(n,K,T) \ &\leq\ \frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i} EC\left[F_{A(T-i)}^{-1}\left(F_{S^c}(nK)\right), T-i\right] \\ &\quad -e^{-\delta T}\left[nK - F_{S^c}^{-1}\left(F_{S^c}(nK)\right)\right]\left(1 - F_{S^c}(nK)\right). \end{aligned}$$

Until now, we assumed that $T-n+1 > 0$. We will now turn to the case that $T-n+1 \leq 0$. Then we know the prices $A(T-n+1), A(T-n+2), \cdots, A(0)$, and only the prices $A(1), \cdots, A(T)$ remain random. Therefore we obtain:

$$\begin{aligned} AC(n,K,T) &= \frac{e^{-\delta T}}{n} E^Q\left[\left(\sum_{i=0}^{n-1} A(T-i) - nK\right)_+\right] \qquad (61) \\ &= \frac{e^{-\delta T}}{n} E^Q\left[\left(\sum_{i=0}^{T-1} A(T-i) - \left(nK - \sum_{i=T}^{n-1} A(T-i)\right)\right)_+\right] \end{aligned}$$

Under this assumption we can apply the same method as above in order to obtain upper bounds for the price of the Asian option. Now we define S^c by

$S^c = \sum_{i=0}^{T-1} F^{-1}_{A(T-i)}(U)$. For $F^{-1+}_{S^c}(0) < nK - \sum_{i=T}^{n-1} A(T-i) < F^{-1}_{S^c}(1)$, we obtain

$$AC(n,K,T) \leq \frac{1}{n} \sum_{i=0}^{T-1} e^{-\delta i}\, EC\left[F^{-1(\alpha)}_{A(T-i)}\left(F_{S^c}\left(nK - \sum_{i=T}^{n-1} A(T-i)\right)\right), T-i\right], \tag{62}$$

where α is determined by

$$F^{-1(\alpha)}_{S^c}\left[F_{S^c}\left(nK - \sum_{i=T}^{n-1} A(T-i)\right)\right] = nK - \sum_{i=T}^{n-1} A(T-i). \tag{63}$$

Note that a similar procedure can be used to derive upper bounds for the price of arithmetic Asian put options.

Using the theory explained in Section T.5.3, we can also derive lower bounds for the price of Asian options. This will be illustrated in the next section.

5.3 Application in a Black & Scholes Setting

In the model of Black & Scholes (1973), the price of the risky asset is described by a stochastic process $\{A(t),\ t \geq 0\}$ following a geometric Brownian motion with constant drift μ and constant volatility σ:

$$\frac{dA(t)}{A(t)} = \mu dt + \sigma d\overline{B}(t), \qquad t \geq 0, \tag{64}$$

with initial value $A(0) > 0$, and where $\left\{\overline{B}(t),\ t \geq 0\right\}$ is a standard Brownian motion.

Under the equivalent martingale measure Q, the price process $\{A(t),\ t \geq 0\}$ still follows a geometric Brownian motion, with the same volatility but with drift equal to the continuously compounded risk-free interest rate δ:

$$\frac{dA(t)}{A(t)} = \delta dt + \sigma dB(t), \qquad t \geq 0, \tag{65}$$

with initial value $A(0)$, and where $\{B(t),\ t \geq 0\}$ is a standard Brownian motion in the Q-dynamics. Hence, under the equivalent martingale measure, we have that

$$A(t) = A(0)\, e^{\left(\delta - \frac{\sigma^2}{2}\right)t + \sigma B(t)}, \qquad t \geq 0. \tag{66}$$

This implies that under the equivalent martingale measure, the random variables $\frac{A(t)}{A(0)}$ are lognormally distributed with parameters $\left(\delta - \frac{\sigma^2}{2}\right) t$ and $t\sigma^2$ respectively:

$$F_{A(t)}(x) = \Pr\left[A(0)e^{(\delta-\frac{\sigma^2}{2})t+\sqrt{t}\sigma\Phi^{-1}(U)} \leq x\right], \tag{67}$$

where U is uniformly distributed on the interval $(0, 1)$.

From (T.78) and (T.80), we find

$$\begin{aligned} EC(K,T) &= e^{-\delta T} E^Q\left[(A(T) - K)_+\right] \\ &= A(0)\ \Phi\left(d_1\right) - K\ e^{-\delta T}\ \Phi\left(d_2\right), \end{aligned} \tag{68}$$

where d_1 and d_2 are given by

$$d_1 = \frac{\ln\left(A(0)/K\right) + (\delta + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{69}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{70}$$

This formula is the famous Black & Scholes (1973) pricing formula for European call options.

Within the Black & Scholes model, no closed form expression is available for the price of an arithmetic Asian call option. Therefore, we will derive upper and lower bounds for the price of such options. We will only consider the case that the averaging has not yet started. The other case can be dealt with in a similar way.

From (56) and (T.82), we find the following comonotonic upper bound for the price of an Asian call option:

$$\begin{aligned} AC(n,K,T) &\leq \frac{e^{-\delta T}}{n} E\left[(S^c - nK)_+\right] \\ &= \frac{A(0)}{n}\sum_{i=0}^{n-1} e^{-\delta i}\ \Phi\left[\sigma\sqrt{T-i} - \Phi^{-1}\left(F_{S^c}(nK)\right)\right] \\ &\quad -e^{-\delta T}\ K\ \left(1 - F_{S^c}(nK)\right), \end{aligned} \tag{71}$$

which holds for any $K > 0$. Note that this upper bound corresponds to the optimal linear combination of the prices of European call options as mentioned in the previous section.

The remaining problem is how to calculate $F_{S^c}(nK)$. The latter quantity follows from

$$\sum_{i=0}^{n-1} F_{A(T-i)}^{-1}\left(F_{S^c}(nK)\right) = nK,$$

or, equivalently, from (66) and Theorem T.1 we find that $F_{S^c}(nK)$ follows from

$$A(0)\sum_{i=0}^{n-1} \exp\left[\left(\delta - \frac{\sigma^2}{2}\right)(T-i) + \sigma\sqrt{T-i}\ \Phi^{-1}\left(F_{S^c}(nK)\right)\right] = nK. \quad (72)$$

Lower bounds for $AC(n,K,T)$ can be obtained from Section T.5.3. Therefore, consider the conditioning random variable Λ defined by

$$\Lambda = \sum_{j=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2}\right)(T-j)} B(T-j). \quad (73)$$

From (66) we find that, in the Q-dynamics,

$$\sum_{i=0}^{n-1} A(T-i) = A(0)\ \sum_{i=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2}\right)(T-i)+\sigma B(T-i)}. \quad (74)$$

Hence, Λ is a linear transformation of a first order approximation to $\sum_{i=0}^{n-1} A(T-i)$. The variance of Λ is given by

$$\sigma_\Lambda^2 = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2}\right)(2T-j-k)} \min\left(T-j, T-k\right). \quad (75)$$

We have that $(B(T-n+1),\ B(T-n+2), \ldots,\ B(T))$ has a multivariate normal distribution. This implies that given $\Lambda = \lambda$, the random variable $B(T-i)$ is normally distributed with mean $r_{T-i}\frac{\sqrt{T-i}}{\sigma_\Lambda}\lambda$ and variance $(T-i)\left(1-r_{T-i}^2\right)$ where

$$r_{T-i} = \frac{cov\left(B(T-i),\Lambda\right)}{\sigma_\Lambda\ \sqrt{T-i}} = \frac{\sum_{j=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2}\right)(T-j)} \min\left(T-i, T-j\right)}{\sigma_\Lambda\ \sqrt{T-i}}. \quad (76)$$

We find that

$$
\begin{aligned}
S^l &= E^Q\left[\sum_{i=0}^{n-1} A(T-i) \mid \Lambda\right] \\
&= A(0) \sum_{i=0}^{n-1} e^{\left(\delta-\frac{\sigma^2}{2} r_{T-i}^2\right)(T-i)+\sigma\ r_{T-i}\ \sqrt{T-i}\ \Phi^{-1}(U)} \qquad (77)
\end{aligned}
$$

where U is uniformly distributed on the unit interval. From this expression, we see that S^l is a comonotonic sum of lognormal random variables. Hence, from Section T.5.3 and (T.82), we find the following lower bound for the price of the Asian call option:

$$
\begin{aligned}
AC(n,K,T) &\geq \frac{e^{-\delta T}}{n} E^Q\left[\left(S^l - nK\right)_+\right] \\
&= \frac{A(0)}{n} \sum_{i=0}^{n-1} e^{-\delta i}\ \Phi\left[\sigma\ r_{T-i}\ \sqrt{T-i} - \Phi^{-1}\left(F_{S^l}(nK)\right)\right] \\
&\quad -e^{-\delta T}\ K\ \left(1 - F_{S^l}(nK)\right) \qquad (78)
\end{aligned}
$$

which holds for any $K > 0$. In this case, $F_{S^l}(nK)$ follows from

$$
A(0) \sum_{i=0}^{n-1} \exp\left[\left(\delta - \frac{\sigma^2}{2} r_{T-i}^2\right)(T-i) + \sigma\ r_{T-i}\sqrt{T-i}\ \Phi^{-1}\left(F_{S^l}(nK)\right)\right] = nK. \qquad (79)
$$

When the number of averaging dates n equals 1, the Asian call option reduces to a European call option. It is straightforward to prove that in this case the upper and the lower bounds (71) and (78) for the price of the Asian option both reduce to the Black & Scholes formula for the price of the European call option.

5.4 Numerical illustration

In this section we numerically illustrate the bounds for the price of Asian options in a Black & Scholes setting, as obtained in the previous section. We consider a time unit of one day. The parameters that were used to generate the results given in Tables 6, 7 and 8 are the same as in Jacques (1996): an initial stock price $A(0) = 100$, a risk-free interest rate of 9% per year, three values (0.2, 0.3 and 0.4) for the yearly volatility, and five values (80, 90, 100,

σ	K	LB	UB	MC (s.e.)
0.2	80	21.9212	21.9269	21.9233 (0.0468)
	90	12.6768	12.7204	12.6714 (0.0432)
	100	5.4609	5.5557	5.4726 (0.0329)
	110	1.6252	1.7072	1.6114 (0.0183)
	120	0.3317	0.3673	0.3336 (0.0080)
0.3	80	22.2332	22.2720	22.2651 (0.0684)
	90	13.8521	13.9512	13.8473 (0.0609)
	100	7.4787	7.6229	7.4395 (0.0484)
	110	3.4826	3.6214	3.5405 (0.0346)
	120	1.4125	1.5105	1.4003 (0.0219)
0.4	80	22.9646	23.0525	22.9694 (0.0880)
	90	15.3589	15.5115	15.3927 (0.0788)
	100	9.5113	9.7041	9.5987 (0.0665)
	110	5.4794	5.6720	5.5574 (0.0517)
	120	2.9608	3.1222	2.9519 (0.0377)

Table 6: Lower (LB) and upper (UB) bounds for the price of an Asian option with $T = 120$ and $n = 30$, compared to Monte Carlo estimates (MC) and their standard error (s.e.).

110 and 120) for the exercise price K. Note that the risk-free force of interest per day is given by $\delta = \frac{\ln(1.09)}{365}$, while the daily volatility σ is obtained by dividing the yearly volatility by $\sqrt{365}$.

In Table 6 we compare the upper and lower bounds (71) and (78) with Monte Carlo estimates (based on 50000 paths each) in case $T = 120$ and $n = 30$. Note that the quasi-random paths are based on antithetic variables in order to reduce the variance of the Monte Carlo estimate and that we generated different paths for each value of σ and K. We also computed the standard error for each estimate. As is well-known, the (asymptotic) 95% confidence interval is given by the estimate plus or minus 1.96 times the standard error. On the other hand, the range between the lower bound and the upper bound contains the exact price with certainty.

Despite the quite large number of paths (and consequently a long computing time) and the variance reduction technique used, the 95% confidence interval is wider than the [LB,UB]-interval in 10 cases out of 15. This indicates that the bounds should be preferred over simulation in this case.

σ	K	LB	UB	MC (s.e.)
0.2	80	20.7841	20.7845	20.7839 (0.0297)
	90	11.0273	11.0599	11.0205 (0.0287)
	100	3.2013	3.3443	3.1984 (0.0196)
	110	0.3373	0.4080	0.3383 (0.0064)
	120	0.0116	0.0185	0.0128 (0.0011)
0.3	80	20.8122	20.8268	20.8055 (0.0441)
	90	11.4929	11.6017	11.5160 (0.0410)
	100	4.5063	4.7221	4.4711 (0.0289)
	110	1.1516	1.3134	1.1458 (0.0150)
	120	0.1915	0.2503	0.1945 (0.0059)
0.4	80	20.9708	21.0309	20.9719 (0.0581)
	90	12.2468	12.4384	12.2183 (0.0514)
	100	5.8157	6.1038	5.8711 (0.0393)
	110	2.2082	2.4582	2.2224 (0.0248)
	120	0.6783	0.8223	0.6802 (0.0135)

Table 7: Lower (LB) and upper (UB) bounds for the price of an Asian option with $T = 60$ and $n = 30$, compared to Monte Carlo estimates (MC) and their standard error (s.e.).

Moreover, the Monte Carlo estimate exceeds the lower bound 6 times. This might indicate that the lower bound is very close to the real price. The upper bound appears to perform better the more the option is in the money.

In Table 7 we use the same parameters as in Table 6 but we change the expiration time to $T = 60$. Now, the 95% confidence interval is wider than the [LB,UB]-interval in 6 cases, but the Monte Carlo estimate exceeds the lower bound 7 times. So again, the lower bound must be very close to the real price.

In Table 8 we change the expiration time back to $T = 120$ but we reduce the number of averaging days to $n = 10$. With these parameters, simulation performs really bad since the simulated confidence interval is wider than the real confidence interval in all cases. The Monte Carlo estimate again exceeds the lower bound 7 times.

The upper bound performs better for the option with $n = 10$ than for the options with $n = 30$. This illustrates the fact that the dependency structure of the $A(T-i)$ is more "comonotonic-like" if all points in time $T-i$ are close

σ	K	LB	UB	MC (s.e.)
0.2	80	22.1712	22.1735	22.1718 (0.0495)
	90	13.0085	13.0232	13.0219 (0.0460)
	100	5.8630	5.8934	5.8793 (0.0351)
	110	1.9169	1.9442	1.9411 (0.0211)
	120	0.4534	0.4665	0.4517 (0.0098)
0.3	80	22.5656	22.5795	22.5524 (0.0720)
	90	14.3149	14.3475	14.2825 (0.0644)
	100	8.0101	8.0563	8.0009 (0.0522)
	110	3.9475	3.9928	3.9788 (0.0382)
	120	1.7297	1.7633	1.7322 (0.0250)
0.4	80	23.4194	23.4493	23.4137 (0.0933)
	90	15.9549	16.0045	15.9191 (0.0833)
	100	10.1735	10.2354	10.1853 (0.0705)
	110	6.1019	6.1643	6.0895 (0.0563)
	120	3.4683	3.5220	3.4844 (0.0431)

Table 8: Lower (LB) and upper (UB) bounds for the price of an Asian option with $T = 120$ and $n = 10$, compared to Monte Carlo estimates (MC) and their standard error (s.e.).

to each other.

6 Conclusions

In this paper, we demonstrated the usefulness of the concept of comonotonicity for describing dependencies between random variables in several financial and actuarial applications. We showed that very tight upper bounds as well as lower bounds can be obtained using the techniques described in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002). It is shown how the techniques can be used to determine provisions for future payment obligations, taking into account the stochastic nature of the return process.

We also demonstrated that the same techniques based on the concept of comonotonicity can be used to derive tight bounds for the price of an arithmetic Asian option, which is essentially a stop-loss premium of a sum of strongly positively dependent random variables.

The upper bounds are especially sharp in case the random components of a sum are rather strongly positive dependent, as they are in many actuarial applications of a financial nature, where the consecutive summands contain a stochastic discounting component. On the other hand, the lower bounds perform very well even in a situation where the dependencies are not strongly positive.

The tightness of the lower and upper bounds together with their easy computability makes them useful instruments for tackling several actuarial problems occuring in practice.

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References

[1] Black, F.; Scholes, M. (1973). "The pricing of options and corporate liabilities", *Journal of Political Economy*, 81 (May-June), 637-659.

[2] Cox, J.C.; Ross, S.A.; Rubinstein, M. (1979). "Option pricing: a simplified approach", *Journal of Financial Economics*, 7, 229-263.

[3] De Schepper, A.; Teunen, M.; Goovaerts, M.J. (1994). "An analytical inversion of a Laplace transform related to annuities certain", *Insurance: Mathematics & Economics*, 14(1), 33-37.

[4] Dhaene, J.; Denuit, M.; Goovaerts, M.J.; Kaas, R.; Vyncke, D. (2001). "The concept of comonotonicity in actuarial science and finance: theory", *Insurance: Mathematics & Economics*, accepted.

[5] Dufresne, D. (1990). "The distribution of a perpetuity with applications to risk theory and pension funding", *Scandinavian Actuarial Journal*, 9, 39-79.

[6] Embrechts, P.; Mc.Neil, A.; Straumann, D. (2001). "Correlation and dependency in risk management: properties and pitfalls", in "Risk Management: Value at Risk and Beyond", edited by Dempster, M. and Moffatt, H.K., Cambridge University Press.

[7] Gerber, H.U.; Shiu, E.S.W. (1996). "Actuarial bridges to dynamic hedging and option pricing", *Insurance: Mathematics & Economics*, 18, 183-218.

[8] Goovaerts, M.J.; Dhaene, J;. De Schepper, A. (2000). "Stochastic Upper Bounds for Present Value Functions", *Journal of Risk and Insurance Theory*, 67.1, 1-14.

[9] Harrison, J.; Kreps, D. (1979). " Martingales and arbitrage in multiperiod securities markets", *Journal of Economic Theory, 20*, 381-408.

[10] Harrison, J.; Pliska, R. (1981). "Martingales and stochastic integrals in the theory of continuous trading", *Stochastic Processes and their Applications, 11*, 215-260.

[11] Jacques, M. (1996). "On the Hedging Portfolio of Asian Options", *ASTIN Bulletin, 26*, 165-183.

[12] Kaas, R.; Dhaene, J.; Goovaerts, M.J. (2000). "Upper and lower bounds for sums of random variables", *Insurance: Mathematics & Economics* 23, 151-168.

[13] Kaas, R.; Goovaerts, M.J.; Dhaene, J.; Denuit, M. (2001). *Modern Actuarial Risk Theory*, Kluwer Academic Publishers, Dordrecht.

[14] Kemna, A.G.Z.; Vorst, A.C.F. (1990). " A pricing method for options based on average asset values", *Journal of Banking and Finance,* 14, 113-129.

[15] Milevsky, M.A. (1997). "The present value of a stochastic perpetuity and the Gamma distribution", *Insurance: Mathematics & Economics*, 20(3), 243-250.

[16] Rogers, L.C.G.; Shi, Z. (1995). "The value of an Asian option", *Journal of Applied Probability* 32, 1077-1088.

[17] Vazquez-Abad, F.J.; Dufresne, D. (1998). "Accelerated simulation for pricing Asian options", *Research Paper Nr 62, Centre for Actuarial Studies, The University of Melbourne.*