The Clique Partitioning Problem: Facets and Patching Facets

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The clique partitioning problem (CPP) can be formulated as follows: Given is a complete graph $G = (V, E)$, with edge weights $w_{ij} \in \mathbb{R}$ for all $\{i, j\} \in E$. A subset $A \subseteq E$ is called a clique partition if there is a partition of V into nonempty, disjoint sets $V_1, \ldots, V_k$, such that each V_p ($p = 1, \ldots, k$) induces a clique (i.e., a complete subgraph), and $A = \cup_{p=1}^{k} \{\{i, j\} | i, j \in V_p, i \neq j\}$. The weight of such a clique partition A is defined as $\sum_{\{i,j\} \in A} w_{ij}$. The problem is now to find a clique partition of maximum weight. The clique partitioning polytope P is the convex hull of the incidence vectors of all clique partitions of G. In this paper, we introduce several new classes of facet-defining inequalities of P. These suffice to characterize all facet-defining inequalities with right-hand side 1 or 2. Also, we present a procedure, called patching, which is able to construct new facets by making use of already-known facet-defining inequalities. A variant of this procedure is shown to run in polynomial time. Finally, we give limited empirical evidence that the facet-defining inequalities presented here can be of use in a cutting-plane approach for the clique partitioning problem. © 2001 John Wiley & Sons, Inc.

Keywords: polyhedral combinatorics; clique partitioning; facets

1. INTRODUCTION

The problem of partitioning a set of objects into disjoint groups while optimizing some measure of the inter- and intragroup relationships is a basic problem in combinatorial optimization. In this paper, we rely on a graph-theoretic interpretation of this problem, called the *clique partitioning problem* (CPP), which can be described as follows: Given is a complete graph $G = (V, E)$ with edge weights $w_{ij} \in \mathbb{R}$ for all $\{i, j\} \in E$. A subset $A \subseteq E$ is called a clique partition if there is a partition of V into nonempty disjoint sets $V_1, \ldots, V_k$ such that

$$A = \bigcup_{p=1}^{k} \{\{i, j\} | i, j \in V_p, i \neq j\}.$$

The weight of such a clique partition is defined as $\sum_{\{i,j\} \in A} w_{ij}$. The problem is now to find a clique partition of maximum weight. Mathematically, CPP can be formulated as follows:

$$\begin{aligned} &\text{Maximize} && \sum_{\{i,j\} \in E} w_{ij} x_{ij} && \\ &\text{subject to} && x_{ij} + x_{ir} - x_{jr} \leq 1 && \text{for all distinct } i, j, r \in V \\ &&& x_{ij} \in \{0, 1\} && \text{for all } \{i, j\} \in E. \end{aligned} \tag{1}$$

If $x_{ij} = 1$, then edge $\{i, j\}$ is in the clique partition; otherwise, it is not. The inequalities in the linear program (1) are called *triangle inequalities*. The clique partitioning polytope P is defined as the convex hull of all feasible solutions to (1).

Knowledge of the facial structure of P is a prerequisite for solving instances of CPP using cutting planes. In a computational study by Grötschel and Wakabayashi [13], it is shown that for a particular set of real-life instances the triangle inequalities usually suffice to arrive at an integral solution. A natural question to ask is whether this phenomenon holds for other classes of real-life instances as well. In Subsection 1.1, we introduce a problem in *group technology* and describe how this problem gives rise to instances of CPP. Next, we give empirical evidence that for this type of instances of CPP, in

Received 1 December 1998; accepted August 1, 2001
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most cases, the triangle inequalities alone will not suffice to arrive at an integral solution. This observation is one reason for our search for valid and facet-defining inequalities.

Another reason is that the CPP is a "basic" clustering problem. Indeed, all kinds of variants of formulation (1) are possible by adding constraints: One can, for instance, add cardinality constraints on the size of the cliques and/or one can specify an upper bound on the number of cliques. Notice, however, that for these variants inequalities that are valid for P are also valid for the polytopes corresponding to such a variant. Thus, the inequalities presented here may also be of use in a wider context than the CPP. Further, observe that the so-called *multicut polytope* (MC) is isomorphic to the clique partitioning polytope P. Indeed, any multicut can be written as $\mathbf{1} - x, x \in P$, and MC is the convex hull of all vectors $\mathbf{1} - x, x \in P$. MC was studied in Deza et al. [8].

The paper is organized as follows: Section 2 deals with new facets of P. In Subsection 2.1, we present three classes of valid inequalities for P, and for each class, we show that under certain conditions these inequalities define facets of P. These classes are generalizations of inequalities described by Grötschel and Wakabayashi [14] and by Müller and Schulz [23]. In Section 2.2, we describe how facet-defining inequalities can be lifted. In Section 3, we present a technique which tries to make use of already-known facet-defining inequalities. This technique, which we call *patching*, takes as input two or more facet-defining inequalities and delivers, if certain conditions are met, a new facet-defining inequality. Similar approaches have been proposed by Grötschel and Wakabayashi [15] and for the TSP by Queyranne and Wang [27]. Not only can we use patching to increase knowledge concerning the facial structure of P, it is also conceivable that patching can be used when separating fractional solutions (see Section 3). Further, in Section 4, we prove that the inequalities described so far allow us to characterize all facet-defining inequalities with right-hand side 1 or 2. Section 5 is devoted to the complexity of separating inequalities. We show that it is NP-hard to decide whether there exists a violated (S, T)-inequality with fixed $|S|$. This motivates the heuristic approach employed to search for violated (S, T)-inequalities with $|S| = 1$. We also show that a variant of the patching procedure can be implemented to run in polynomial time. Finally, in Section 6, we indicate, using real-life instances from group technology, the potential usefulness of the inequalities presented here. Section 7 contains the conclusion.

Related Literature

Applications of the CPP were described in Marcotorchino and Michaud [20], Ali and El-Rewini [2], Grötschel and Wakabayashi [13], and Kolen and Lenstra [17]. Branch-and-bound procedures were proposed in Palubeckis [26] and Dorndorf and Pesch [9]. In Bandelt et al. [3], the CPP was studied for arbitrary (not necessarily complete) graphs. A number of studies dealt with related problems. Chopra and Rao [6] studied the partition problem, which differs from CPP in the sense that the number of subsets of vertices is prespecified, and the graph G is not required to be complete. Crama and Oosten [7] studied the special case where the graph G is required to be bipartite and is to be partitioned into complete bipartite subgraphs. Further, Faigle et al. [10] dealt with a partitioning problem where the number of vertices in each subset is bounded.

1.1. Motivation

The following problem is of crucial importance in group technology (see, for instance, Ham et al. [16]). Consider a set of p parts to be produced by a set of q machines. Given is a $0-1 p \times q$ matrix A, with $a_{ij} = 1$ if and only if part i has to visit machine j in order to complete processing ($i = 1, \dots, p, j = 1, \dots, q$). Consider now a partition of the set of parts and machines into so-called *cells*. If in such a cell there is a part i and a machine j such that $a_{ij} = 0$, the pair (i, j) is called a *void*. If there is a part i and a machine j such that $a_{ij} = 1$ while they are not in a same cell, we call the pair (i, j) an *exception*. The problem is now to find a partition that minimizes the number of voids and exceptions. Of course, different versions of this problem exist (one can impose a bound on the number of cells or on the number of items within a cell, or one can give weights to voids and exceptions, etc); however, here we will only consider the version described above.

Let us now describe how an instance of CPP arises from this problem. Construct a complete graph $G = (V, E)$ in the following way: There is a vertex for each part (a "part-vertex") and there is a vertex for each machine (a "machine-vertex"), so $|V| = p + q$. An edge between two part-vertices (machine-vertices) has weight 0, and an edge between a part-vertex (corresponding to part i) and a machine-vertex (corresponding to machine j) has weight $+1$ if $a_{ij} = 1$ and -1 if $a_{ij} = 0$. This completely specifies an instance of CPP.

As mentioned in the Introduction, Grötschel and Wakabayashi [13] presented a cutting-plane algorithm using triangle inequalities and so-called (S, T)-inequalities (see Section 2 for a description of these inequalities). To separate this latter class of inequalities, they used a heuristic procedure. We implemented their algorithm, selected from the literature seven well-known, medium-sized group-technology instances, and ran the algorithm. The results are depicted in Table 1. The first column describes the group-technology instance and its source, the second column gives its size, the third column (denoted by z_{LP}) gives the value of the LP-relaxation [where (i) denotes integrality of the solution], and the final column gives the value of the objective function after running the algorithm of Grötschel and Wakabayashi (z_{GW}). It turns out that in all cases except one no integral solution was found.

1.2. Notation

If $G = (V, E)$ is a (not necessarily complete) graph, and $S, T \subseteq V$ are two nonempty sets, then we denote the set of edges in G with one vertex in S and the other in T by $E(S, T)$, that is:

$$E(S,T) = \{\{i, j\} \in E | i \in S, j \in T\}.$$

We write $E(S)$ for $E(S, S)$.

If $F \subseteq E$, then $\chi^F \in \{0,1\}^{|E|}$ denotes the incidence vector (also called characteristic vector) of F, and $V(F)$ denotes the set of vertices incident to an edge in F. If $x \in \mathbb{R}^{|E|}$, then we define

$$x(F) = \sum_{\{i,j\} \in F} x_{ij}.$$

The *support* of an inequality $a^T x \leq a_0$ is the graph $H = (V(F), F)$ induced by the edges in $F = \{\{i, j\} \in E | a_{ij} \neq 0\}$. Analogously, the *positive support* of an inequality $a^T x \leq a_0$ is the graph $H^+ = (V(F^+), F^+)$ induced by the edges in $F^+ = \{\{i, j\} \in E | a_{ij} > 0\}$.

2. FACETS OF *P*

We assume that the reader is familiar with the fundamentals of polyhedral theory (see, e.g., Nemhauser and Wolsey [25]). Therefore, we introduce only some basic terminology without going into details.

A set $P \subseteq \mathbb{R}^n$ is called a polyhedron if P is the intersection of finitely many half-spaces, that is:

$$P = \{x \in \mathbb{R}^n | Ax \leq b\},$$

for some $m \times n$ matrix A and some $b \in \mathbb{R}^m$. If P is bounded, then P can also be written as the convex hull of finitely many points in $\mathbb{R}^n$, and P is called a polytope. The dimension of P, denoted by $\dim(P)$, is the maximum number of affinely independent points in P minus 1. If, for some $a \in \mathbb{R}^n, a \neq 0, a_0 \in \mathbb{R}, P$ is contained in the half-space $\{x \in \mathbb{R}^n | a^T x \leq a_0\}$, then the inequality $a^T x \leq a_0$ is said to be valid for P. The set $F = P \cap \{x \in \mathbb{R}^n | a^T x = a_0\}$ is called a face of P. A face F is called a proper face if $\emptyset \neq F \neq P$. Notice that F itself is a polyhedron (or a polytope), so we can refer to the dimension of F. If F is a proper face of P, then $\dim(F) \leq \dim(P) - 1$; if $\dim(F) = \dim(P) - 1$, then F is called a facet, and the inequality $a^T x \leq a_0$ is said to be facet-defining. Facet-defining inequalities are necessary in the linear description of P.

Grötschel and Wakabayashi [14] showed that the triangle inequalities define facets of the clique partitioning polytope corresponding to a complete graph $G = (V, E)$,

TABLE 1. Results of a cutting plane algorithm.

Instance	Size	z_{LP}	z_{GW}
KKV [18]	9×15	23.0000 (i)	—
SUL [30]	20×11	48.0000	46.5833
SEI [29]	11×22	55.6667	54.1964
LES [19]	13×25	42.0000	36.6256
MCC [21]	24×16	56.6667	51.1740
BOC [4]	16×43	82.0000	75.3628
GRO [12]	23×20	75.3333	68.5182

as well as the lower-bound constraints $x_{ij} \geq 0$ $(i, j \in V)$ in the LP-relaxation of (1). They also proved that the following three classes of inequalities define facets of the clique partitioning polytope corresponding to a complete graph:

(i) Let $S, T \subseteq V$ be two nonempty disjoint subsets of V. Then, the inequality

$$x(E(S,T)) - x(E(S)) - x(E(T)) \leq \min\{|S|, |T|\} \quad (2)$$

is valid. It defines a facet if and only if $|S| \neq |T|$. It is called a 2-partition inequality or an (S, T)-inequality.

(ii) Let $U = \{u_0, \ldots, u_{k-1}\} \subseteq V, k \geq 5$, and let $C = \{\{u_i, u_{i+1}\} | i = 0, \ldots, k-1\}, \bar{C} = \{\{u_i, u_{i+2}\} | i = 0, \ldots, k-1\}$ (indices modulo k). Then, the inequality

$$x(C) - x(\bar{C}) \leq \left\lfloor \frac{|C|}{2} \right\rfloor \quad (3)$$

is valid. It defines a facet if and only if k is odd, and it is called a 2-chorded cycle inequality.

(iii) Let $U = \{u_0, \ldots, u_{k-1}\} \subseteq V, k \geq 3, z \in V \backslash U$, and let $P = \{\{u_i, u_{i+1}\} | i = 0, \ldots, k-2\}, \bar{P} = \{\{u_i, u_{i+2}\} | i = 0, \ldots, k-3\}, R = \{\{z, u_i,\} | i = 0, \ldots, k-1, i \text{ odd}\}, \bar{R} = \{\{z, u_i,\} | i = 0, \ldots, k-1, i \text{ even}\}$. Then, the inequality

$$x(P \cup R) - x(\bar{P} \cup \bar{R}) \leq \left\lfloor \frac{|U|}{2} \right\rfloor \quad (4)$$

is valid. It defines a facet if and only if $|U|$ is odd, and it is called a 2-chorded path inequality. If $u_{k-1} = u_1, u_k = u_2$, and $|U| = k - 2 \geq 8$ is even, then (4) is also valid and facet-defining and it is called a 2-chorded wheel inequality.

Notice that *a priori* it is not clear that an inequality which is facet-defining for the polytope corresponding to a complete graph on n vertices is also facet-defining for the polytope corresponding to a complete graph on $m \geq n$ vertices. However, a result in Bandelt et al. [3] implies that this is the case.

In the remainder of this section, we describe two different generalizations of (S, T)-inequalities with $|S| = 1$, and we give sufficient conditions for these inequalities to be facet-defining. We also describe a generalization of the 2-chorded cycle inequalities. Our proofs showing that these inequalities are facet-defining are based on standard techniques and use the type of arguments that are used, for instance, in [14]: We show that any facet containing the face induced by the inequality under consideration is, in fact, identical to this face. Hence, the inequality must be facet-defining. More specifically, this is done by comparing feasible solutions that satisfy the inequality at equality and deducing the value of coefficients in the inequality. We refer the interested reader to the thesis of Rutten [28].

2.1. *New Classes of Facet-defining Inequalities*

2.1.1. Weighted (*s*, *T*)-Inequalities Let $G = (V, E)$ be a complete graph and consider the following inequalities:

$$c \cdot x(E(\{s\}, T)) - x(E(T)) \leq \binom{c+1}{2}, \qquad (5)$$

with $c \in \mathbb{N}, s \in V, T \subseteq V \backslash \{s\}$. Figure 1 shows the structure of inequalities (5). These inequalities were introduced by Müller and Schulz [23] in a more general setting. We refer to inequalities (5) as *weighted* (s, T)-*inequalities*. Notice that for $c = 1$ the weighted (s, T)-inequalities reduce to 2-partition inequalities induced by $\{s\}$ and T.

We have the following theorem (see Rutten [28]):

Theorem 2.1. Let $c \geq 2$. Then, the weighted (s, T)-inequalities (5) are valid for P, and they define facets if and only if $|T| \geq c + 2$.

The necessity of the condition in Theorem 2.1 was also shown in Müller and Schulz [24] within their more general context.

2.1.2. Stable Set Inequalities The 2-partition inequalities can also be generalized in a way different from the one described in Theorem 2.1, as we show in the sequel.

If $G = (V, E)$ is an arbitrary graph, then $\alpha(G)$ denotes the *stability number* of G, that is, the maximum cardinality of a stable (i.e., independent) set of G, and $k(G)$ denotes the *clique cover number* of G, that is, the size of a smallest possible clique cover. Moreover, E is said to be *minimal* with respect to $\alpha(G)$ if removing any edge from E increases $\alpha(G)$.

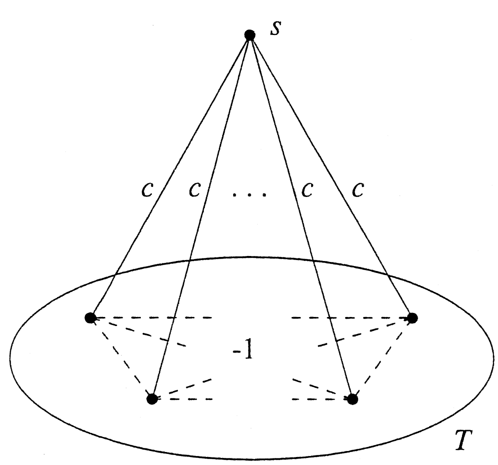

FIG. 1. Support of a weighted (*s*, *T*)-inequality.

Let $G = (V, E)$ be a complete graph and consider the following inequalities:

$$x(E(\{s\}, T)) - x(F) \leq \alpha(H), \qquad (6)$$

with $F \subseteq E, T = V(F) \neq V, s \in V \backslash T$, and $H = (T, F)$. We refer to inequalities (6) as *stable set inequalities*. Notice that if F is a clique a stable set inequality reduces to a 2-partition inequality.

We have the following theorem (see Rutten [28]):

Theorem 2.2. The stable set inequalities (6) are valid for P. The following conditions are necessary conditions for these inequalities to be facet-defining:

(i) F is minimal with respect to $\alpha(H)$,
(ii) $k(H) = \alpha(H) = 1$ or $k(H) > \alpha(H) \geq 2$,
(iii) H is connected,
(iv) For all $i \in T$, there is a stable set of size $\alpha(H)$ containing i, and
(v) For all $i \in T$, there is a stable set of size $\alpha(H)$ not containing i.

Conditions (i)–(v), together with the following condition, are sufficient for H to ensure that (6) is facet-defining:

(vi) For all $i, j \in T, \{i, j\} \notin F$, there is a stable set $S \subseteq T$ of cardinality $\alpha(H)$ such that $S \cap \{i, j\} = \emptyset$.

Remark. Inequalities (6) generalize the so-called wheel inequalities found by Chopra and Rao [6] [the wheel inequalities occur when $H = (T, F)$ is an odd cycle]. Figure 2 shows two graphs which both satisfy conditions (i)–(vi) and which are, therefore, facet-inducing graphs for inequalities of the form (6).

2.1.3. Generalized 2-Chorded Cycle Inequalities In the beginning of this section, we described the 2-chorded cycle inequalities. In this subsection, we show that 2-chorded cycles plus some additional 3-chords (with coefficient 1) and some 4-chords (with coefficient -1) also define facets of the clique partitioning polytope. We need the following notation:

Let $U = \{u_0, \ldots, u_{k-1}\} \subseteq V, k \geq 6$, and let

$$\begin{aligned} C &= \{\{u_i, u_{i+1}\} | i = 0, \ldots, k-1\}, \\ \bar{C} &= \{\{u_i, u_{i+2}\} | i = 0, \ldots, k-1\}, \\ D &\subseteq \{\{u_i, u_{i+3}\} | i = 0, \ldots, k-1\}, \\ \bar{D} &= \bigcup_{\{u_i, u_{i+3}\} \in D} \{\{u_{i-1}, u_{i+3}\}, \{u_i, u_{i+4}\}\} \end{aligned} \qquad (7)$$

(indices modulo k), such that no two edges in D have a common vertex and $D \cap \bar{D} = \emptyset$. (Note that, by definition, $D \cap \bar{D} = \emptyset$ for $k \geq 8$, since D consists of 3-chords and $\bar{D}$ consists of 4-chords.) So, C is a cycle of length $k, \bar{C}$ is the set of 2-chords of C, D is a set of 3-chords, and $\bar{D}$ is a set of (induced) 4-chords (see Fig. 3 for an example).

Consider now the following inequalities:

$$x(C \cup D) - x(\bar{C} \cup \bar{D}) \leq \lfloor \frac{1}{2} k \rfloor. \qquad (8)$$

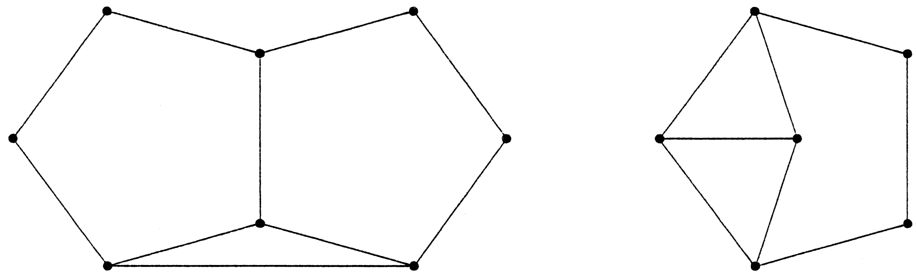

FIG. 2. Facet-inducing graphs for inequality (6).

We refer to these inequalities as *generalized* 2-*chorded cycle inequalities*. We have the following theorem (see Rutten [28]):

Theorem 2.3. The generalized 2-chorded cycle inequalities (8) are valid for P. They define facets if

(i) $D = \{\{u_i, u_{i+3}\}\}, k \geq 9, k$ odd, or
(ii) $D = \{\{u_i, u_{i+3}\}, \{u_{i+1}, u_{i+4}\}\}, k \geq 9, k$ odd, or
(iii) $D = \{\{u_i, u_{i+3}\}, \{u_{i+1}, u_{i+4}\}, \{u_{i+2}, u_{i+5}\}\}, k \geq 7, k$ odd,

for any $u_i \in U$.

Remark. Let $D_j \subseteq \{\{u_i, u_{i+3}\} | i = 1, \ldots, k\}$ $(j = 1, 2, \ldots)$ satisfy one of the conditions of Theorem 2.3, and let $\bar{D}_j$ be defined as in (7). Then, also, for $D = \cup_j D_j$ and $\bar{D} = \cup_j \bar{D}_j$, inequality (8) defines a facet of P, provided that no edge in $D_j \cup \bar{D}_j$ has a vertex in common with an edge in $D_l \cup \bar{D}_l$ for any $j \neq l$.

2.2. *Lifting*

In the previous subsection, we described several classes of facet-defining inequalities for the clique partitioning polytope. In this subsection, we describe a lifting technique that can be used to transform facet-defining inequalities into new facet-defining inequalities.

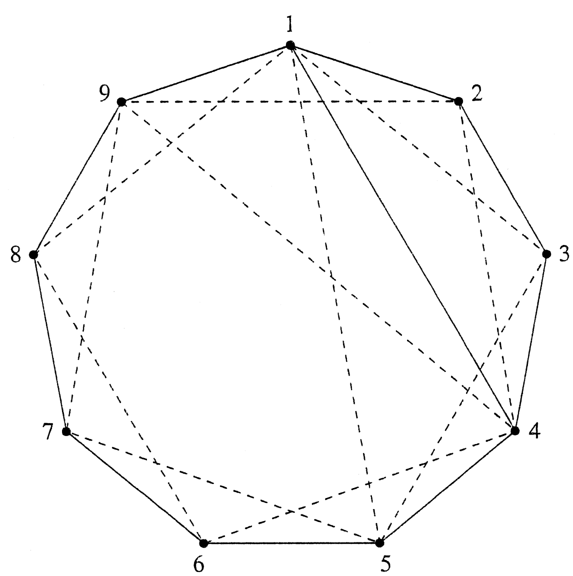

FIG. 3. Support of a generalized 2-chorded cycle inequality.

Let $H = (U, F)$ be a subgraph of the graph $G = (V, E)$ (i.e., $U \subseteq V$ and $F \subseteq E$). If $a^T x \leq a_0$ defines a facet of the clique partitioning polytope corresponding to H, then by sequentially lifting the coefficients of the edges in $E \backslash F$, we find a facet-defining inequality of the clique partitioning polytope corresponding to G (see Wolsey [31] and Nemhauser and Wolsey [25]). This is stated more precisely in the following lemma:

Lemma 2.4 (sequential lifting [31])**.** Let $G = (V, E)$ be a complete graph, $F \subseteq E$, and let $a^T x \leq a_0$ be a facet-defining inequality for the polytope

$$P \cap \{x \in \mathbb{R}^{|E|} | x_{ij} = 0 \text{ for all } \{i, j\} \in F\}.$$

Let $f = \{u, v\} \in F$, and define $\bar{a} \in \mathbb{R}^{|E|}$ by

$$\bar{a}_{ij} = \begin{cases} a_{ij} & \text{if } \{i,j\} \in E \backslash F \\ 0 & \text{if } \{i,j\} \in F \backslash \{f\} \\ a_0 - \max\{a^T x | x \in P, x_{uv} = 1, x_{ij} = 0 \\ \qquad \forall \{i,j\} \in F \backslash \{f\}\} \\ \quad \text{if } \{i,j\} = \{u,v\}. \end{cases} \tag{9}$$

Then, the inequality $\bar{a}^T x \leq a_0$ defines a facet of the polytope

$$P \cap \{x \in \mathbb{R}^{|E|} | x_{ij} = 0 \text{ for all } \{i, j\} \in F \backslash \{f\}\}.$$

By repeatedly applying Lemma 2.4, one can derive new facets of the clique partitioning polytope from known facets. In general, the optimization problem in (9) is difficult to solve; however, in some special cases (see Lemma 2.5), it can be quite easy.

Sometimes, lifting a valid inequality which is not facet-defining results in a facet-defining inequality. For instance, if $F = \delta(v)$ $(v \in V)$, then lifting the upper-bound constraint $x_{ij} \leq 1$ $(i, j \in V \backslash \{v\})$, which, in general, does not define a facet, results in a triangle inequality, no matter in what order the coefficients are lifted. However, in general, lifting the coefficients of the edges in F in different orders results in different facets.

Let us refer to an inequality where one vertex (called the *center*) is incident to all edges with positive coefficients as a *star-inequality*. Weighted (s, T)-inequalities (which include the (S, T)-inequalities with $|S| = 1$) and stable set inequalities are examples of star-inequalities.

Lemma 2.5 ([28]). Let $G = (V, E)$ be a complete graph, $V = \{s\} \cup T_1 \cup T_2$, and let $a^T x \leq a_0$ be a facet-defining start inequality of the clique partitioning polytope corresponding to $G' = (\{s\} \cup T_1, E(\{s\} \cup T_1))$ with the property that $a_{ij} > 0$ if and only if $\{i, j\} \in E(\{s\}, T_1)$. Then, the inequality $\bar{a}^T x \leq a_0$ defined by

$$\bar{a}_{ij} = \begin{cases} a_{ij} & \text{for all } \{i,j\} \in E(\{s\} \cup T_1) \\ a_0 & \text{for all } \{i,j\} \in E(\{s\}, T_2) \\ -a_0 & \text{for all } \{i,j\} \in E(T_2) \\ -a_{si} & \text{for all } i \in T_1, j \in T_2 \end{cases}$$

is facet-defining for the clique partitioning polytope corresponding to G.

Inequalities satisfying the condition of Lemma 2.5 are the weighted (s, T)-inequalities and the stable set inequalities. Hence, we have the following two corollaries:

Corollary 2.6. Let $G = (V, E)$ be a complete graph, $\{s\}, T_1, T_2 \subseteq V$ be disjoint sets, and $c \geq 2$. Then, the inequality

$$c \cdot x(E(\{s\}, T_1)) + \binom{c+1}{2} \cdot x(E(\{s\}, T_2)) - x(E(T_1)) - \binom{c+1}{2} \cdot x(E(T_2)) - c \cdot x(E(T_1, T_2)) \leq \binom{c+1}{2} \quad (10)$$

defines a facet of the clique partitioning polytope.

Proof. This follows from Theorem 2.1 and Lemma 2.5. ∎

We refer to these inequalities as *lifted weighted* (s, T)*-inequalities*. Notice that for each number $n \in \mathbb{N}$ there is a lifted weighted (s, T)-inequality containing a variable with coefficient n and containing a variable with coefficient $-n$.

Corollary 2.7. Let $G = (V, E)$ be a complete graph, $\{s\}, T_1, T_2 \subseteq V$ be disjoint sets, and $H = (T_1, F)$ be a graph satisfying conditions (i)–(vi) of Theorem 2.2. Then, the inequality

$$x(E(\{s\}, T_1)) - x(F) + \alpha(H) \cdot x(E(\{s\}, T_2)) - \alpha(H) \cdot x(E(T_2)) - x(E(T_1, T_2)) \leq \alpha(H) \quad (11)$$

defines a facet of the clique partitioning polytope.

Proof. This follows from Theorem 2.2 and Lemma 2.5. ∎

We refer to these inequalities as *lifted stable set inequalities*.

Next, we consider a generalization of the 2-chorded path and the 2-chorded wheel inequalities.

Let $G = (V, E)$ be a complete graph, $U_0 \subseteq V, U_1 = \{u_1, \ldots, u_k\} \subseteq V \backslash U_0$, and let

$$\begin{aligned} P &= \{\{u_i, u_{i+1}\} | i = 1, \ldots, k-1\}, \\ \bar{P} &= \{\{u_i, u_{i+2}\} | i = 1, \ldots, k-2\}, \\ R &= \{\{u_0, u_i\} | u_0 \in U_0, i = 1, \ldots, k, i \text{ even}\}, \\ \bar{R} &= \{\{u_0, u_i\} | u_0 \in U_0, i = 1, \ldots, k, i \text{ odd}\}, \end{aligned} \quad (12)$$

that is, P is a path of length $k-1$, $\bar{P}$ is the set of 2-chords of this path, and R and $\bar{R}$ are sets of edges from U_0 to U_1. We have the following theorem (see Rutten [28]):

Theorem 2.8. Let $P, \bar{P}, R, \bar{R}$ be as described by (12). Then, the inequality

$$x(P \cup R) - x(\bar{P} \cup \bar{R}) - x(E(U_0)) \leq \lfloor \frac{1}{2} k \rfloor \quad (13)$$

is valid for the clique partitioning polytope corresponding to G. If k is odd, then (13) defines a facet of this polytope.

A similar result can be obtained for the 2-chorded wheel inequalities (see Rutten [28]):

Theorem 2.9. Let $G = (V, E)$ be a complete graph, $U_0 \subseteq V, U_1 = \{u_1, \ldots, u_k\} \subseteq V \backslash U_0, u_{k-1} = u_1, u_k = u_2, |U_1| = k-2$ even, and let $P, \bar{P}, R, \bar{R}$ be as described by (12). Then, the inequality

$$x(P \cup R) - x(\bar{P} \cup \bar{R}) - x(E(U_0)) \leq \frac{1}{2} |U_1| \quad (14)$$

is valid and facet-defining for the clique partitioning polytope corresponding to G.

3. PATCHING PROCEDURES

It is often possible to use a set of valid inequalities to construct a new valid inequality that is not implied by the former. Well-known examples are the procedures to derive Gomory–Chvatal cuts or so-called vertex cloning or edge-contraction techniques. In this section, we focus on so-called *patching procedures*. A *patching procedure* combines a set of inequalities into a single one such that the nonzero coefficients of these inequalities remain unchanged. Notice that a patching procedure can be used within a cutting plane algorithm, without *a priori* establishing (using patching) a set of inequalities which could contain a violated inequality. More concrete, given a fractional solution, one could conceive of a procedure that determines two (or more) "almost violated" inequalities and next patch these two inequalities into a single (facet-defining) inequality. In this way, the fractional solution is used to guide the patching procedure to construct inequalities that have a relatively high likelihood of being violated.

Grötschel and Wakabayashi [15] presented some patching procedures for (S, T)-inequalities. One of their procedures can informally be described as follows: Take nonempty subsets S_1, S_2, T_1, and T_2 of V, pairwise disjoint, with the exception of the intersection T of T_1 and T_2. Assume that $|T| \geq 2, |S_1| \leq |T_1 \backslash T|$, and $|S_2| \leq |T_2 \backslash T|$. Then, the facet-defining (S, T)-inequalities

$$x(E(S_1, T_1)) - x(E(S_1)) - x(E(T_1)) \leq |S_1|$$

and

$$x(E(S_2, T_2)) - x(E(S_2)) - x(E(T_2)) \leq |S_2|$$

can be combined to

$$\begin{aligned} &x(E(S_1, T_1)) + x(E(S_2, T_2)) - x(E(S_1 \cup S_2)) \\ &\quad -x(E(T_1)) - x(E(T_2)) + x(E(T)) \leq |S_1| + |S_2|, \end{aligned} \tag{15}$$

which is also facet-defining (see [15]). Notice that compared to just adding up the two (S, T)-inequalities the coefficients of the variables in $E(T)$ and $E(S_1, S_2)$ are raised and lowered, respectively, by one.

Here, we restrict ourselves to combining pairs of inequalities and we distinguish two cases of patching, depending on the intersection—referred to as V_I—of the vertex sets of the supports of the inequalities involved. If this intersection V_I is empty, we refer to combining the two inequalities as *disjoint patching*; otherwise, we call it *intersection patching*. Notice that in the case of intersection patching the two inequalities must be such that the two coefficients corresponding to any edge induced by V_I are identical. The value of the right-hand side of the new inequality equals the sum of the right-hand sides of the original inequalities minus a correction for the edges induced by the vertex set V_I. To determine the value of this correction, we introduce the concept of a *covering*. Intuitively, a covering equals the maximal contribution of the edges induced by V_I to the left-hand side.

Definition 3.1. A covering of a vertex set V_C with respect to an inequality $a^T x \leq a_0$ is denoted as $\text{cov}_a[V_C]$ and defined as follows:

$$\begin{aligned} \text{cov}_a[V_C] = a_0 - \max_{x \in \{0,1\}^{|E|}} \{a^T x | x_{ij} = 0 \\ \text{for all } \{i, j\} \in E(V, V_C), x \text{ feasible}\}. \end{aligned}$$

As an illustration, consider the triangle inequality $x_{12} + x_{13} - x_{23} \leq 1$. It is easy to verify that the covering of any vertex set containing vertex 1 equals 1. Notice that the covering of any vertex set V_C is nonnegative, assuming that the inequality is valid. Further, the covering of a single vertex s equals zero if and only if there exists a clique partition not incident to s satisfying $a^T x \leq a_0$ at equality. Finally, the definition implies that the following inequalities are valid:

$$\sum_{\{i,j\} \in E(V \backslash V_C)} a_{ij} x_{ij} \leq a_0 - \text{cov}_a[V_C], \tag{16}$$

$$\sum_{\{i,j\} \in E(V_C)} a_{ij} x_{ij} \leq \text{cov}_a[V_C]. \tag{17}$$

Based on this concept we can make an interesting observation that seems to suggest that the patching approach may be well suited for investigating the facial structure of the clique partitioning polytope. Intuitively, we show that it is possible to use "small" facet-defining inequalities as the core of larger facet-defining inequalities. More specifically, given some valid inequality $\sum_{\{i,j\} \in E(V)} \pi_{ij} x_{ij} \leq \pi_0$, suppose that for some $V_C \subseteq V$ the inequality $\sum_{\{i,j\} \in E(V_C)} \pi_{ij} x_{ij} \leq \text{cov}_\pi[V_C]$ defines a facet. In the following lemma, we show, then, that there exists a facet-defining inequality $\delta^T x \leq \delta_0$, implying that $\pi^T x \leq \pi_0$ with $\delta_{ij} = \beta \pi_{ij}$ for all $\{i, j\} \in E(V_C)$.

Lemma 3.2. *Basic patching principle.*

Let V_C be an arbitrary nonempty subset of V, and E_0, a (possibly empty) arbitrary subset of $E(V_C, V \backslash V_C)$. Let P^* be the face of the clique partitioning polytope P defined by the equalities $x_{ij} = 0$ for all $\{i, j\} \in E_0$. Given is a valid inequality $\pi^T x \leq \pi_0$, defining a proper face of P^*, called F_π.

Assume that the valid inequality $\rho^T x = \sum_{\{i,j\} \in E(V_C)} \pi_{ij} x_{ij} \leq \text{cov}_\pi[V_C]$ defines a nontrivial facet of P. Then, there exists a facet $F = \{x \in P^* | \delta^T x = \delta_0\}$ of P^*, containing F_π, and there is a $\beta > 0$ such that $\delta_{ij} = \beta \pi_{ij}$ for all $\{i, j\} \in E(V_C)$.

Proof. The inequality $\rho^T x \leq \text{cov}_\pi[V_C]$ defines a facet, so there must exist a nonzero π_{ij}, for some $\{i, j\} \in E(V_C)$. It follows, since $\pi^T x \leq \pi_0$ defines a proper face of P^*, that there exists a facet of P^*, containing F_π, defined by some inequality $\delta^T x \leq \delta_0$ for which δ_{ij} is nonzero.

To proceed, take a maximal affinely independent set of feasible integral solutions satisfying $\rho^T x \leq \rho_0 := \text{cov}_\pi[V_C]$ at equality. Let M be the $\dim(P) \times \dim(P)$ matrix such that there is precisely one column for every solution of the set, and each column represents the incidence vector of the corresponding solution. Then, $\rho^T M = \rho_0 \cdot \mathbf{1}_{|E(V)|}$, where $\mathbf{1}_{|E(V)|}$ is a row vector with $|E(V)|$ elements, all at value 1.

Let ρ_C and x_C be vectors with the elements of ρ and x, respectively, that correspond to the edges in $E(V_C)$. There exist $|E(V_C)|$ columns such that the rows in these columns corresponding to the elements of $E(V_C)$ form a matrix M_C that is nonsingular. Since the coefficients of ρ corresponding to edges covering vertices not in V_C are all zero, we have $\rho_C^T M_C = \rho_0 \cdot \mathbf{1}_{|E(V_C)|}$, implying that

$$\rho_C^T = \rho_0 \cdot \mathbf{1}_{|E(V_C)|} \cdot M_C^{-1}. \tag{18}$$

Notice that $\rho_0 > 0$ since $\rho^T x \leq \rho_0$ is assumed to define a nontrivial facet.

Now, let y be a feasible solution such that $\pi^T y = \pi_0 - \text{cov}_\pi[V_C]$, and $y_{ij} = 0$ for all $\{i, j\} \in E(V, V_C)$. Such a solution always exists due to the definition of a covering.

Then, for each solution x corresponding to a column of M_C, we construct a solution z as follows:

$$z_{ij} = \begin{cases} x_{ij} & \text{if } \{i,j\} \in E(V_C), \\ y_{ij} & \text{if } \{i,j\} \in E(V \backslash V_C), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that z is in P^*. Further, it is easy to verify that for each solution z we have $\pi^T z = \rho_0 + \pi_0 - \text{cov}_\pi[V_C] = \pi_0$. In other words, when $\tilde{M}$ refers to an $|E(V)| \times |E(V_C)|$ matrix where each column represents the incidence vector of a solution z, we have $\pi^T \tilde{M} = \pi_0 \cdot \mathbf{1}_{|E(V_C)|}$, and since F contains F_π, $\delta^T \tilde{M} = \delta_0 \cdot \mathbf{1}_{|E(V_C)|}$.

The construction of the solutions z implies that the rows in $\tilde{M}$ corresponding to the elements of $E(V_C)$ form the nonsingular matrix M_C. Notice further that for all columns in $\tilde{M}$ the part of each column not in M_C is identical. Let δ_C be the vector with the elements of δ that correspond to the edges in $E(V_C)$. We derive $\delta^T \tilde{M} = \delta_C^T M_C + \gamma \cdot \mathbf{1}_{|E(V_C)|} = \delta_0 \cdot \mathbf{1}_{|E(V_C)|}$, where γ is the constant resulting from the identical parts of the columns. Notice that $0 \le \gamma \le \delta_0$; otherwise, solutions could be constructed violating the inequality $\delta^T x \le \delta_0$. So,

$$\delta_C^T = (\delta_0 - \gamma)\mathbf{1}_{|E(V_C)|} M_C^{-1}. \tag{19}$$

Since δ_C is nonzero, it follows that $\delta_0 - \gamma \ne 0$. Equalities (18) and (19) imply that $\delta_{ij} = [(\delta_0 - \gamma)/\rho_0]\pi_{ij}$ for all $\{i,j\} \in E(V_C)$. This completes the proof. ∎

Remark. Notice that arguments used in the proof of this theorem imply that *for each* facet of P^* containing F_π, induced by some inequality $\omega^T x \le \omega_0$, there exists a *nonnegative* coefficient ϕ such that $\omega_{ij} = \phi \pi_{ij}$ for all $\{i,j\} \in E(V_C)$.

To describe the intersection patching procedure, we make the following assumptions:

(i) Let $a^T x \le a_0$ and $b^T x \le b_0$ define different facets of P, with V_a (V_b) the vertex set of the support of $a^T x \le a_0$ ($b^T x \le b_0$),
(ii) $V_a \cap V_b \ne \emptyset$. Let $V_I := (V_a \cap V_b) \cup (V \backslash (V_a \cup V_b))$,
(iii) P^* is the face of P defined by the equalities $x_{ij} = 0$ for all $\{i,j\} \in E(V_a \backslash V_I, V_b \backslash V_I)$,
(iv) $a_{ij} = b_{ij}$ for all $\{i,j\} \in E(V_I)$,
(v) $\text{cov}_a[V_I] = \text{cov}_b[V_I]$, and
(vi) $\text{cov}_a[V^0] + \text{cov}_b[V_I \backslash V^0] \ge \text{cov}_a[V_I]$ for all $V^0 \subseteq V_I$.

Theorem 3.3. *Intersection patching.*
The inequality

$$\sum_{\{i,j\} \in E(V_a)} a_{ij} x_{ij} + \sum_{\{i,j\} \in E(V_b, V_b \backslash V_I)} b_{ij} x_{ij} \le a_0 + b_0 - \text{cov}_a[V_I] \tag{20}$$

defines a facet of P^* unless $a_{ij} = b_{ij} = 0$ for all $\{i,j\} \in E(V_I)$ and $\text{cov}_a[V_I] = \text{cov}_b[V_I] = 0$.

Proof. First, we show that the new inequality is valid for P^*. Let x be an arbitrary integer solution in P^*. Then, it is possible to partition V_I into two sets $V_I(a)$ and $V_I(b)$, such that every vertex $b \in V_I$ that is in the same clique as a vertex in $V_b \backslash V_I$, is in $V_I(b)$, and all other vertices in V_I are in $V_I(a)$. Then, we have, using assumptions (v) and (vi) and inequality (16),

$$\begin{aligned} &\sum_{\{i,j\} \in E(V_a)} a_{ij} x_{ij} + \sum_{\{i,j\} \in E(V_b, V_b \backslash V_I)} b_{ij} x_{ij} \\ &\quad = \sum_{\{i,j\} \in E(V_a, V_a \backslash V_I(b))} a_{ij} x_{ij} + \sum_{\{i,j\} \in E(V_b, V_b \backslash V_I(a))} b_{ij} x_{ij} \\ &\quad \le a_0 - \text{cov}_a[V_I(b)] + b_0 - \text{cov}_b[V_I(a)] \\ &\quad \le a_0 + b_0 - \text{cov}_a[V_I]. \end{aligned}$$

This shows that inequality (20) is valid.

Now, we show that it is facet-defining. Let us invoke Lemma 3.2, where we set $E_0 = E(V_a \backslash V_I, V_b \backslash V_I)$, $V_C = V_I \cup V_a$ and where $\pi^T x \le \pi_0$ is represented by inequality (20) and $\rho^T x \le \rho_0$ is represented by $a^T x \le a_0$. Lemma 3.2 implies that there exists a facet of P^* defined by the inequality $\delta^T x \le \delta_0$, containing the face induced by inequality (20), such that

$$\delta_{ij} = \alpha a_{ij} \text{ for all } \{i,j\} \in E(V_I \cup V_a) \text{ for some } \alpha > 0. \tag{21}$$

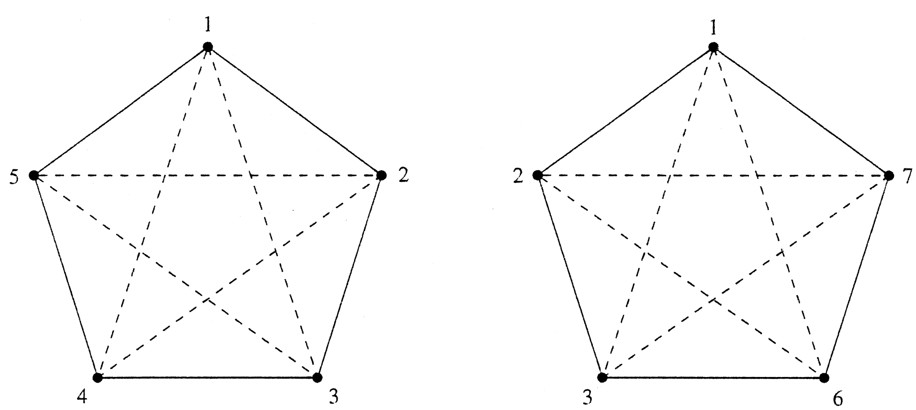

FIG. 4. Two odd-cycle inequalities.

Now, let us again apply Lemma 3.2 where we set $E_0 = E(V_a \backslash V_I, V_b \backslash V_I), V_C = V_I \cup V_b$ and where $\pi^T x \le \pi_0$ is represented by inequality (20) and $\rho^T x \le \rho_0$ is represented by $b^T x \le b_0$. As discussed in the remark following the proof of Lemma 3.2, it follows that

$$\delta_{ij} = \beta b_{ij} \text{ for all } \{i, j\} \in E(V_I \cup V_b) \text{ for } \beta \ge 0. \quad (22)$$

We now distinguish two cases:

(i) There exists an $\{i', j'\} \in E(V_I)$ such that $a_{i',j'} \ne 0$. We have $\delta_{i'j'} = \alpha a_{i'j'} = \beta b_{i'j'}$ for all $\{i', j'\} \in E(V_I)$. Since $a_{i'j'} = b_{i'j'} \ne 0$, it follows that $\alpha = \beta > 0$. Thus, by (21) and (22), the coefficients δ_{ij} equal the coefficients of inequality (20) up to multiplication with a strictly positive scalar which implies that inequality (20) is facet-defining for P^*.

(ii) There exists no $\{i, j\} \in E(V_I)$ such that $a_{ij} \ne 0$.

First, consider a solution satisfying inequality (20) and satisfying $a^T x \le a_0$ at equality. Such a solution exists by the definition of a covering. We derive, using equalities (21) and (22),

$$\begin{aligned} \delta_0 &= \sum_{\{i,j\} \in E(V)} \delta_{ij} x_{ij} = \sum_{\{i,j\} \in E(V_I \cup V_a)} \alpha a_{ij} x_{ij} \\ &\quad + \sum_{\{i,j\} \in E(V_b \backslash V_I)} \beta b_{ij} x_{ij} \\ &= \alpha a_0 + \beta(b_0 - \text{cov}_b[V_I]). \end{aligned}$$

Next, consider a solution satisfying inequality (20) and inequality $b^T x \le b_0$ at equality. In a similar fashion, we derive $\delta_0 = \alpha(a_0 - \text{cov}_a[V_I]) + \beta b_0$. Combining these equalities, we obtain $\alpha \text{ cov}_a[V_I] = \beta \text{ cov}_b[V_I]$. If $\text{cov}_a[V_I] > 0$, it follows that $\alpha = \beta > 0$. Thus, the coefficients δ_{ij} equal the coefficients of inequality (20) up to multiplication with a scalar, which implies that inequality (20) is facet-defining for P^*.

In the case $\text{cov}_a[V_I] = 0$, the new inequality is just the sum of the two facet-defining inequalities $a^T x \le a_0$ and $b^T x \le b_0$ and, therefore, not facet-defining. ∎

As described in Section 2.2, facet-defining inequalities of P^* can be lifted to facet-defining inequalities of P by applying ordinary sequential lifting procedures.

Example 3.4. Consider the following two odd-cycle inequalities for an instance of clique partitioning on seven vertices (see Fig. 4). Let us refer to the first inequality as $a^T x \le a_0$:

$$x_{12} + x_{23} + x_{34} + x_{45} + x_{51} - x_{13} - x_{24} - x_{35} - x_{41} - x_{52} \le 2,$$

$$x_{12} + x_{23} + x_{36} + x_{67} + x_{71} - x_{13} - x_{26} - x_{37} - x_{61} - x_{72} \le 2.$$

It follows that $V_I = \{1, 2, 3\}$ and $\text{cov}_a[V_I] = 1$. Notice that assumptions (i)–(vi) are satisfied. Thus, by Theorem 3.3, the inequality

$$x_{12} + x_{23} + x_{34} + x_{45} + x_{51} + x_{36} + x_{67} + x_{71} - x_{13} - x_{24}$$
$$- x_{35} - x_{41} - x_{52} - x_{26} - x_{37} - x_{61} - x_{72} \le 2 + 2 - 1 = 3$$

defines a facet of $\{x \in P | x_{57} = x_{56} = x_{47} = x_{46} = 0\}$. By sequentially lifting the variables x_{57}, x_{56}, x_{47}, and x_{46} (in that order!), we find the inequality (see Fig. 5)

$$x_{12} + x_{23} + x_{34} + x_{45} + x_{51} + x_{36} + x_{67} + x_{71} - x_{13} - x_{24}$$
$$- x_{35} - x_{41} - x_{52} - x_{26} - x_{37} - x_{61} - x_{72} - x_{57} \le 3,$$

which defines a facet of P.

It is not difficult to verify that the intersection patching procedure can be applied to any two odd-cycle inequalities such that the positive support of V_I is a path with an odd number of vertices.

Sometimes, the derivation of a facet of P from a facet of P^* is straightforward, and an explicit formulation can be given. Consider the case of patching two star-inequalities:

Corollary 3.5. *Star patching.*

Let $a^T x \le 1$ be an (S, T)-inequality with $S = \{s\}$ and let $b^T x \le c$ be a facet-defining star inequality, such that their supports have only the center s in common. Then, the inequality

$$\sum_{\{i,j\} \in E(V)} \gamma_{ij} x_{ij} \le c$$

is facet-defining, where

$$\gamma_{ij} = \begin{cases} c \cdot a_{ij} & \text{if } \{i, j\} \in E(V_a), \\ b_{ij} & \text{if } \{i, j\} \in E(V_b), \\ -b_{js} & \text{if } \{i, j\} \in E(V_a \backslash \{s\}, V_b \backslash \{s\}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Notice that for a star inequality the covering of the center equals the right-hand side. Applying Theorem 3.3 and then sequentially lifting the coefficients corresponding to the edges in $E(V_a \backslash \{s\}, V_b \backslash \{s\})$ yield the corollary. ∎

To describe the disjoint patching procedure, we make the following assumptions:

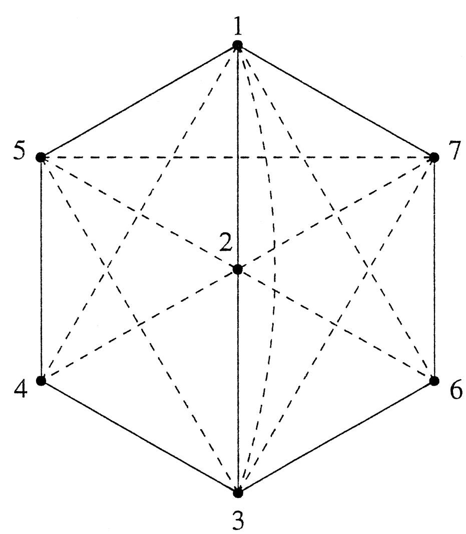

FIG. 5. Patching of two odd-cycle inequalities.

(i) Let $a^T x \le a_0$ and $b^T x \le b_0$ define different facets of P, with V_a (V_b) the vertex set of the support of $a^T x \le a_0$ ($b^T x \le b_0$),
(ii) $V_a \cap V_b = \emptyset$,
(iii) q, r, and s are three different vertices in V_a,
(iv) t, u, and v are three different vertices in V_b,
(v) P^* is the face of P defined by the equalities $x_{ij} = 0$ for all $\{i, j\} \in E(V_a, V_b) \setminus E(\{q, r, s\}, \{t, u, v\})$, and
(vi) $\gamma_{st} := \text{cov}_a[s] = \text{cov}_b[t]$.

Now, to patch the two inequalities, we are looking for values of the coefficients γ_{ij} such that the following inequality defines a facet of P^*:

$$(a + b)^T x + \sum_{\{i,j\} \in E(V_a, V_b)} \gamma_{ij} x_{ij} \le a_0 + b_0.$$

In the next two theorems, we give two options for the values of the coefficients γ_{ij}.

Theorem 3.6. *Disjoint patching, option I* (*Fig. 6*).

Assume that $\text{cov}_a[q] = \text{cov}_a[r] = \text{cov}_b[u] = \text{cov}_b[v] = 0$, $\text{cov}_a[\{q, s\}] = a_{qs}$, $\text{cov}_a[\{r, s\}] = a_{rs}$, $\text{cov}_b[\{u, t\}] = b_{ut}$, and $\text{cov}_b[\{v, t\}] = b_{vt}$. Then, if $\gamma_{st} > 0$, the following inequality defines a facet of P^*:

$$(a + b)^T x + \gamma_{st}(x_{st} - x_{su} - x_{rt} - x_{qv}) \le a_0 + b_0. \quad (23)$$

Proof. First, we show that inequality (23) is valid for P^*. Let x be an arbitrary integral solution in P^*. If s and t are not in the same clique, it follows that $x_{st} = 0$ and inequality (23) is satisfied by the validity of the two original inequalities. So, we may assume that $x_{st} = 1$. Notice that, since x is in P^*, no vertex in $V_a \setminus \{q, r, s\}$ ($V_b \setminus \{t, u, v\}$) can be in a clique with $t(s)$. Now, if neither q nor r are in the same clique as s and t, we have $a^T x \le a_0 - \text{cov}_a[s] = a_0 - \gamma_{st}$ and, therefore, inequality (23) is satisfied. Similarly, if neither u nor v are in the same clique as s and t, we have $b^T x \le b_0 - \gamma_{st}$ and inequality (23) is satisfied. So, either q or r or both are in the same clique as s and t. Also, similarly, either u or v or both are in the same clique as s and t. Now, if r (u) is in the same clique as s and t, then $x_{rt} = 1$ ($x_{us} = 1$) and inequality (23) is satisfied. Otherwise, q and v are in the same clique as s and t, but then $x_{qv} = 1$ and inequality (23) is satisfied.

Now, we show that (23) is facet-defining. Let us invoke Lemma 3.2, where we set $E_0 = E(V_a, V_b) \setminus E(\{q, r, s\}, \{t, u, v\}), V_C = V_a$ and where $\pi^T x \le \pi_0$ is represented by inequality (23) and $\rho^T x \le \rho_0$ is represented by $a^T x \le a_0$. Lemma 3.2 implies that there exists a facet defined by the inequality $\delta^T x \le \delta_0$, containing the face induced by inequality (23), such that

$$\delta_{ij} = \alpha a_{ij} \text{ for all } \{i, j\} \in E(V_a) \text{ for some } \alpha > 0.$$

Now, let us again apply Lemma 3.2 where we set $E_0 = E(V_a, V_b) \setminus E(\{q, r, s\}, \{t, u, v\}), V_C = V_b$ and where $\pi^T x \le \pi_0$ is represented by inequality (23) and $\rho^T x \le \rho_0$ is represented by $b^T x \le b_0$. As discussed in the remark following the proof of Lemma 3.2, it follows that

$$\delta_{ij} = \beta b_{ij} \text{ for all } \{i, j\} \in E(V_b) \text{ for } \beta \ge 0.$$

Summarizing, there exists a facet of P^* containing the face of P^* defined by inequality (23), which is induced by an inequality of the form

$$(\alpha a + \beta b)^T x + \sum_{\{i,j\} \in E(V_a, V_b)} \delta_{ij} x_{ij} \le \alpha a_0 + \beta b_0.$$

Let us now show that α equals β and $\delta_{ij} = \alpha \gamma_{ij}$.

First, consider δ_{ru}. Since $\text{cov}_a[r] = \text{cov}_b[u] = 0$, there is a solution x such that $a^T x = a_0, b^T x = b_0$, and $\{r\}$ and $\{u\}$ are cliques. By changing $\{r\}$ and $\{u\}$ to one clique $\{r, u\}$, we obtain a new solution y satisfying the new inequality at equality. Comparing x and y yields $\delta_{ru} = 0 = \gamma_{ru}$. Analogously, we derive $\delta_{rv} = 0 = \gamma_{rv}$ and $\delta_{qu} = 0 = \gamma_{qu}$.

Next, consider δ_{qt}. Since $\text{cov}_a[q] = 0$ and $\text{cov}_b[\{u, t\}] = b_{ut}$, there is a solution x such that $a^T x = a_0, b^T x = b_0$, and $\{q\}$ and $\{u, t\}$ are cliques. By changing $\{q\}$ and $\{u, t\}$ to one clique $\{q, u, t\}$, we obtain a new solution y satisfying the new inequality at equality. Comparing x and y, we conclude that $\delta_{qt} + \delta_{qu} = 0$. Since $\delta_{qu} = 0$, it follows that $\delta_{qt} = 0 = \gamma_{qt}$. Analogously, one can show that $\delta_{sv} = 0 = \gamma_{sv}$.

Third, consider δ_{st}. Since $\text{cov}_a[\{q, s\}] = a_{qs}$, and by the definition of a covering, there is a solution x such that $a^T x = a_0, b^T x = b_0 - \text{cov}_b[t]$, and $\{q, s\}$ and $\{t\}$ are cliques. By changing $\{q, s\}$ and $\{t\}$ to one clique $\{q, s, t\}$, we obtain a new solution y satisfying the new inequality at equality. Comparing x and y, we conclude that $\delta_{st} = \beta\, \text{cov}_b[t] = \beta \gamma_{st}$. In a similar way, one can derive $\delta_{st} = \alpha\, \text{cov}_a[s] = \alpha \gamma_{st}$. Using (23) and (19), it follows that $\alpha = \beta$.

Finally, consider the negative coefficients. Since $\text{cov}_a[\{q, s\}] = a_{qs}$, and $\text{cov}_b[\{u, t\}] = b_{ut}$, there is a solution x such that $a^T x = a_0, b^T x = b_0$, and $\{q, s\}$ and $\{u, t\}$ are cliques. By changing $\{q, s\}$ and $\{u, t\}$ to one clique $\{q, s, u, t\}$, we obtain a new solution y satisfying the new inequality at equality. Comparing x and y, we have $\delta_{qu} + \delta_{qt} + \delta_{su} + \delta_{st} = 0$. Substituting the earlier results $\delta_{qu} = \delta_{qt} = 0$, it follows that $\delta_{su} = -\delta_{st}$. Analogously, one can show $\delta_{rt} = -\delta_{st}$, and $\delta_{qv} = -\delta_{st}$. ∎

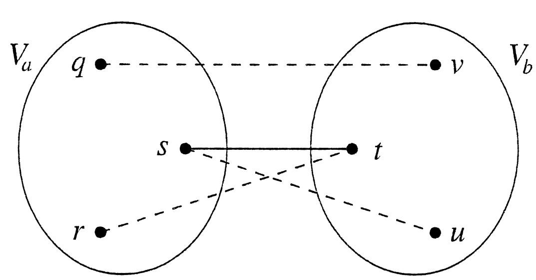

FIG. 6. Disjoint patching according to Option I.

Another possibility for the γ-coefficients is as follows:

Theorem 3.7. *Disjoint patching, option II* (*Fig. 7*).

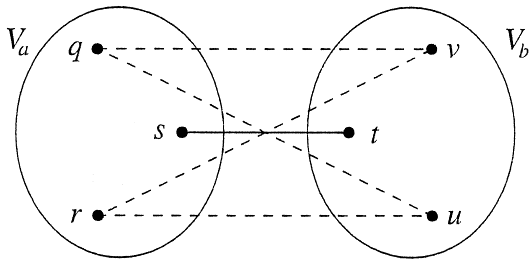

FIG. 7. Disjoint patching according to option II.

Assume that $\text{cov}_a[q] = \text{cov}_a[r] = \text{cov}_b[u] = \text{cov}_b[v] = 0$, $\text{cov}_a[\{q,s\}] = a_{qs}$, $\text{cov}_a[\{r,s\}] = a_{rs}$, $\text{cov}_a[\{q,r,s\}] = a_{qs} + a_{rs} + a_{qr}$, $\text{cov}_b[\{u,t\}] = b_{ut}$, $\text{cov}_b[\{v,t\}] = b_{vt}$, and $\text{cov}_b[\{u,v,t\}] = b_{ut} + b_{vt} + b_{uv}$. Then, if $\gamma_{st} > 0$, the following inequality defines a facet of P^*:

$$(a+b)^T x + \gamma_{st}(x_{st} - x_{qu} - x_{qv} - x_{ru} - x_{rv}) \leq a_0 + b_0. \tag{24}$$

Proof. Similar to the proof of Theorem 3.6, see Rutten [28]. ∎

Starting with the results of Theorem 3.6 and Theorem 3.7, we can derive facets of the full-dimensional polytope by applying ordinary sequential lifting procedures. In some cases, it is possible to derive an explicit formulation of inequalities resulting from a disjoint patching procedure. One example occurs when both inequalities are (S,T)-inequalities with $|S| = 1$.

Theorem 3.8. *Disjoint* (S,T) *patching.*

Let $a^T x \leq 1$ and $b^T x \leq 1$ be (S,T)-inequalities, with in both cases $|S| = 1$, and with disjoint supports on V_a and V_b, respectively. Let $s(t)$ be the center of $a^T x \leq 1$ ($b^T x \leq 1$). Let $V_a = U_a \cup W_a \cup \{s\}$ be such that these three sets are disjoint. Let $V_b = U_b \cup W_b \cup \{t\}$ be such that these three sets are disjoint. Then, the following inequality is valid and facet-defining if U_a and U_b are either both empty or both nonempty and if W_a and W_b are both nonempty:

$$(a+b)^T x + x_{st} - \sum_{j \in U_b} x_{sj} - \sum_{i \in U_a} x_{it} - \sum_{\{i,j\} \in E(W_a,W_b)} x_{ij} \leq 2. \tag{25}$$

Proof. First, we show that the inequality is valid for P. Let x be an arbitrary integral solution in P. If s and t are not in the same clique, the inequality is satisfied by validity of the two original inequalities. So, we may assume that $x_{st} = 1$. If there are no other vertices in V_a in the same clique as s and t, we have $a^T x \leq 0$ and, therefore, inequality (25) is satisfied. Similarly, if there are no other vertices in V_b in the same clique as s and t, we have $b^T x \leq 0$ and inequality (25) is satisfied. So, there is at least one vertex $r \in V_a$ and at least one vertex $u \in V_b$ which are in the same clique as s and t. Then, $r \in U_a, u \in U_b$, or $\{r,u\} \in E(W_a, W_b)$, implying that

$$x_{st} - \sum_{j \in U_b} x_{sj} - \sum_{i \in U_a} x_{it} - \sum_{\{i,j\} \in E(W_a,W_b)} x_{ij} \leq 0,$$

and, therefore, the inequality is satisfied by the validity of the two original inequalities.

Next, we need to show that the inequality defines a facet of P. In case U_a and U_b are both nonempty from Theorem 3.6, and in case U_a and U_b are both empty from Theorem 3.7, we can deduce that there is a facet of P containing the face of P defined by the inequality, which is defined by an inequality of the form

$$\alpha(a+b)^T x + \sum_{\{i,j\} \in E(V_a,V_b)} \delta_{ij} x_{ij} \leq 2\alpha,$$

where δ_{st} equals α. It remains to show that $\delta_{ij} = \alpha\gamma_{ij}$ for all $\{i,j\} \in E(V_a, V_b)$.

First, we consider δ_{ij} for all $\{i,j\} \in E(V_a,V_b) \backslash E(W_a, W_b)$. Since $\text{cov}_a[i] = \text{cov}_b[j] = 0$, there is a solution x such that $a^T x = 1, b^T x = 1$, the new inequality is satisfied at equality, and $\{i\}$ and $\{j\}$ are cliques. Merging $\{i\}$ and $\{j\}$ to one clique $\{i,j\}$, we obtain a new solution y satisfying the new inequality at equality. Comparing x and y, we conclude that $\delta_{ij} = 0 = \gamma_{ij}$.

Next, we consider δ_{sj} for $j \in W_b$. There exists a solution x such that $a^T x = 0, b^T x = 1$, the new inequality is satisfied at equality, and $\{s,t,j\}$ is a clique. Then, x satisfies the patching at equality, yielding $\delta_{st} + \delta_{tj} + \delta_{sj} = 2\alpha$. Plugging in the earlier results $\delta_{st} = \alpha$, and $\delta_{tj} = \alpha$, we have $\delta_{sj} = 0 = \gamma_{sj}$. In a similar way, it is argued that for $i \in W_a$ we have $\delta_{it} = 0 = \gamma_{it}$.

Now, we consider δ_{sj} for $j \in U_b$. There exists a solution x such that $a^T x = 1, b^T x = 1, \{i,s,t,j\}$ is a clique for a $i \in W_a$ (which exists since $W_a \neq \emptyset$) and x satisfies the patching at equality, yielding $\delta_{st} + \delta_{tj} + \delta_{sj} + \delta_{ij} = 0$. Plugging in the earlier results $\delta_{st} = \alpha, \delta_{ij} = 0$, and $\delta_{it} = 0$, we have $\delta_{sj} = -\alpha = \alpha\gamma_{sj}$. In a similar way, it is argued that for $i \in U_a$ we have $\delta_{it} = -\alpha = \alpha\gamma_{it}$.

Finally, we consider δ_{ij} for $\{i,j\} \in E(W_a, W_b)$. There exists a solution x such that $a^T x = 1, b^T x = 1, \{i,s,t,j\}$ is a clique and it satisfies the patching at equality. This yields $\delta_{st} + \delta_{tj} + \delta_{sj} + \delta_{ij} = 0$. Plugging in the earlier results $\delta_{st} = \alpha, \delta_{sj} = 0$, and $\delta_{it} = 0$, we have $\delta_{ij} = -\alpha = \alpha\gamma_{ij}$. ∎

We refer to inequalities (25) as *composed* 2*-partition inequalities* (Fig. 8).

4. FACETS WITH RIGHT-HAND SIDE EQUAL TO 1 OR 2

In this section, we investigate facet-defining inequalities with small right-hand sides. The reason for this is based on the expectation that small right-hand sides give rise to inequalities with a relatively simple structure. This

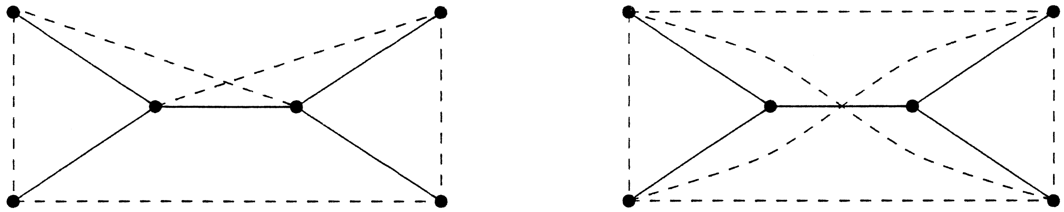

FIG. 8. Disjoint (S, T)-patching of two triangle inequalities.

structure could perhaps be exploited when separating (either exact or heuristically) these inequalities.

4.1. *Right-hand Side Equal to 1*

In Section 2.1, we described the so-called 2-partition inequalities. The following theorem shows that all valid inequalities with the right-hand side equal to 1 are implied by 2-partition inequalities with the right-hand side equal to 1.

Theorem 4.1. Any facet-defining inequality of P with the right-hand side equal to 1 is an (S, T)-inequality with $|S| = 1$.

Proof. We will prove the theorem by showing that any valid inequality with right-hand side 1 is implied by one or more (S, T)-inequalities with $|S| = 1$.

Consider any valid inequality with right-hand side 1. Obviously, the largest coefficient in such a valid inequality equals 1. We are in either one of the following two cases:

(1) There is at most one coefficient in the valid inequality equal to 1. Then, this inequality is trivially implied by $x_{ij} \leq 1$ for some $\{i, j\} \in E$, which, in turn, is implied by two triangle inequalities, which are (S, T)-inequalities with $|S| = 1$ and $|T| = 2$.

(2) There are at least two coefficients in the valid inequality equal to 1. The set of edges corresponding to these coefficients must have a vertex in common, say vertex $i \in V$; otherwise, one can exhibit a clique partition violating the inequality. Consider now a pair of edges, say $\{i, j\}$ and $\{i, k\}$ corresponding to coefficients with value 1. Since $E(\{i, j, k\})$ is a feasible clique partition, the coefficient of $\{j, k\}$ cannot be larger than -1. It is now easy to observe that this valid inequality is implied by an (S, T)-inequality with $S = \{i\}$.

■

This result was independently established in the context of transitive packing by Müller and Schulz [24].

4.2. *Right-hand Side Equal to 2*

To characterize all facets with the right-hand side equal to 2, we first show that the positive support of any facet-defining inequality is connected.

Lemma 4.2. Let $a^T x \leq a_0$ be a facet-defining inequality for P. Then, the positive support (see Section 1.2) of this inequality (referred to as H) is connected.

Proof. Suppose that H is not connected, but instead consists of k components ($k \geq 2$). Let $W_i \subseteq V(F)$ be the set of vertices in component i ($i = 1, \ldots, k$). Since the support of a is connected [14], there exist two components W_i and W_j ($1 \leq i < j \leq k$) and an edge $\{u_i, u_j\} \in E$ with $a_{u_i,u_j} < 0$ joining vertex $u_i \in W_i$ with vertex $u_j \in W_j$. Consider any clique partition A containing this edge, and let

$$B := A \cap \left(\bigcup_{i=1}^{k} E(W_i) \right).$$

Then, on the one hand, all edges $\{i, j\} \in A$ with $a_{ij} > 0$ are also in B, but, on the other hand, $\{u_i, u_j\} \notin B$; hence,

$$a^T \chi^A < a^T \chi^B \leq a_0.$$

So, the face induced by $a^T x = a_0$ is contained in the face induced by $x_{u_i,u_j} = 0$—a contradiction. Thus, H is connected. ■

Now, we can prove the following theorem:

Theorem 4.3. Let $G = (V, E)$ be a complete graph. All facet-defining inequalities of the clique partitioning polytope corresponding to G with right-hand side equal to 2 belong to one of the following classes:

(1) 2-Partition inequalities with right-hand side equal to 2 [see inequalities (2)],
(2) 2-Chorded cycles of length 5 [see inequalities (3) with $k = 5$],
(3) Inequalities obtained by applying intersection patching to two 2-partition inequalities with right-hand side equal to 1 [see inequalities (15) with $|S_1| = |S_2| = 1$],
(4) (Lifted) stable set inequalities [see inequalities (11) with $\alpha(H) = 2$], and
(5) Composed 2-partition inequalities [see inequalities (25)].

Proof. First, observe that the 2-chorded path inequalities with the right-hand side equal to 2 are contained in class (3).

Now, let $a^T x \leq 2$ be a facet-defining inequality. It is obvious that $a_{ij} \leq 2$ for all $\{i, j\} \in E$. We distinguish the following three cases:

(1) There is more than one edge $\{i, j\} \in E$ with $a_{ij} = 2$,
(2) There is exactly one edge $\{i, j\} \in E$ with $a_{ij} = 2$, and
(3) There is no edge $\{i, j\} \in E$ with $a_{ij} = 2$.

(1) In this case, all edges $\{i, j\} \in E$ with $a_{ij} = 2$ are incident to the same vertex, say vertex s. Moreover, all edges $\{i, j\} \in E$ with $a_{ij} = 1$ are also incident to s. For any two edges $\{s, i\}$ and $\{s, j\}$ with coefficient equal to 2 (i.e., $a_{si} = a_{sj} = 2$), we have $a_{ij} \le -2$. Also, if $a_{si} = 2$ and $a_{sj} = 1$, then $a_{ij} \le -1$. Define

$$T = \{i \in V | a_{si} = 1\},$$
$$F = \{\{i, j\} | i, j \in T, a_{ij} \le -1\}.$$

Then, the graph $H = (T, F)$ has a stability number no more than 2; otherwise, the inequality $a^T x \le 2$ would not be valid. Hence, this inequality is implied by a lifted stable set inequality.

(2) Let $\{s, t\}$ be the edge with coefficient equal to 2. Then, all edges with coefficient equal to 1 are either incident to s or to t. Moreover, if $a_{si} = 1$, then $a_{ti} \le -1$, and if $a_{ti} = 1$, then $a_{si} \le -1$. If there are no edges with coefficient equal to 1 incident to s (or to t), then we have the same situation as in case (1). Otherwise, define

$$S = \{i \in V | a_{si} = 1\},$$

$$T = \{i \in V | a_{ti} = 1\},$$

where both S and T are nonempty. If $i, j \in S$, then $a_{ij} \le -1$, and also if $i, j \in T$, we have $a_{ij} \le -1$. Hence, the inequality $a^T x \le 2$ is implied by the sum of the two 2-partition inequalities

$$x(E(\{s\}, S \cup \{t\})) - x(E(S \cup \{t\})) \le 1,$$

$$x(E(\{t\}, T \cup \{s\})) - x(E(T \cup \{s\})) \le 1,$$

and it is therefore not facet-defining.

(3) All edges have coefficient at most 1. Let G' be the graph induced by the edges with coefficient equal to 1. Then, from Lemma 4.2, we know that G' is connected, so it contains a path of length at least 2. On the other hand, $a^T x \le 2$ is a valid inequality, so G' cannot contain a path of length 5 or more (an edge set $P = \{\{v_i, v_{i+1}\} | i = 1, \ldots, l\}$ is called a path if and only if $v_1, \ldots, v_{l+1}$ are distinct, so a cycle of length l contains a longest path of length $l - 1$). We distinguish the following three cases:

(i) A longest path in G' has length 2,
(ii) A longest path in G' has length 3, and
(iii) A longest path in G' has length 4.

(3.i) All edges with a coefficient equal to 1 are incident to the same vertex, say vertex s. Define T and F as in case (1); then, again, the graph $H = (T, F)$ has stability number no more than 2. Hence, the inequality $a^T x \le 2$ is implied by a stable set inequality.

(3.ii) Let $\{v_1, \ldots, v_4\} \subseteq V$, and let $P = \{\{v_i, v_{i+1}\} | i = 1, 2, 3\}$ be a longest path of length 3, where $a_{ij} = 1$ for all $\{i, j\} \in P$. If $a_{v_1,v_4} = 1$, then there are no other edges with coefficient equal to 1, and $a_{v_1,v_3} + a_{v_2,v_4} \le -2$. If any of these two edges has a coefficient of at most -2, then $a^T x \le 2$ is implied by two triangle inequalities. If $a_{v_1,v_3} = a_{v_2,v_4} = -1$, then $a^T x \le 2$ is implied by an (S, T)-inequality with $S = \{v_1, v_3\}$ and $T = \{v_2, v_4\}$. If $a_{v_1,v_4} \le 0$, then all other edges with a coefficient equal to 1 are either incident to v_2 or to v_3. Let $T_1 = \{i \in V | a_{v_2,i} = 1\}$ and $T_2 = \{i \in V | a_{v_3,i} = 1\}$ (hence, $v_1 \in T_1$ and $v_4 \in T_2$). Then, $T_1 \cap T_2 = \emptyset$; otherwise, $a^T x \le 2$ would not be valid. This implies that the inequality $a^T x \le 2$ can be obtained by applying our patching procedure to two (S, T)-inequalities, one with $S = \{v_2\}$ and $T = T_1$ and the other with $S = \{v_3\}$ and $T = T_2$.

(3.iii) Let $\{v_1, \ldots, v_5\} \subseteq V$, and let $P = \{\{v_i, v_{i+1}\} | i = 1, \ldots, 4\}$ be a longest path of length 4, where $a_e = 1$ for all $e \in P$. Observe that $a_{v_1,v_3} \le -1$ and also that $a_{v_3,v_5} \le -1$. If $a_{v_1,v_5} = 1$, then we have a cycle of length 5. All 2-chords of this cycle have coefficient at most -1, and there can be no other edges with a positive coefficient. Hence, the inequality $a^T x \le 2$ is implied by a 2-chorded cycle inequality.

We are left with the situations where $a_{v_1,v_5} \le 0$ (see Fig. 9). All edges with a coefficient equal to 1 not in P are either incident to v_2 or to v_4. In the remainder of the proof, let $T_1 = \{i \in V | a_{v_2,i} = 1\}, T_2 = \{i \in V | a_{v_4,i} = 1\}$ (hence, $v_1, v_3 \in T_1$ and $v_3, v_5 \in T_2$), and $T_0 = T_1 \cap T_2$. If $i, j \in T_k$ $(k = 1, 2)$, then $a_{ij} \le -1$.

First, consider the case where $a_{v_1,v_4} \le 0$ and also $a_{v_2,v_5} \le 0$. If $a_{ij} \le -2$ for all $i, j \in T_0$, then the inequality $a^T x \le 2$ is implied by the sum of two (S, T)-inequalities, one with $S = \{v_2\}$ and $T = T_1$ and the other with $S = \{v_4\}$ and $T = T_2$. If $a_{ij} = -1$ for some $i, j \in T_0$, then $a_{v_2,v_4} \le -1$, and the inequality $a^T x \le 2$ is implied by an intersection patching of the same two (S, T)-inequalities.

If $a_{v_1,v_4} = 1$ or $a_{v_2,v_5} = 1$, then similar arguments show that the inequality $a^T x \le 2$ is either implied by the sum of two (S, T)-inequalities or it is implied by an intersec-

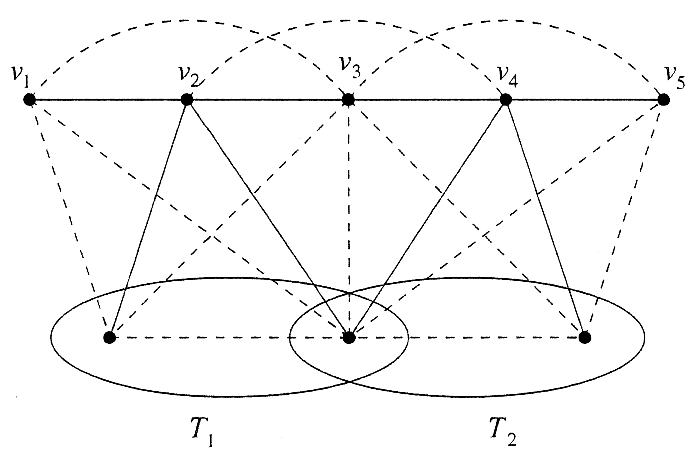

FIG. 9. Support of an inequality with right-hand side 2.

tion patching of two (S,T)-inequalities. This completes the proof of the theorem. ∎

5. THE COMPLEXITY OF SEPARATION

In this section, we consider the complexity of separating inequalities for the clique partitioning polytope. Obviously, when designing an algorithm using polyhedral results, the complexity of the separation problem with respect to certain classes of valid inequalities becomes an important issue. Müller [22] and Caprara and Fischetti [5] showed that the 2-chorded cycle inequalities can be separated in polynomial time. In Subsection 5.1, we show that the question: Given a (fractional) solution x, does there exist an (S,T)-inequality with $|S| = 1$ violated by x? is NP-hard. Indeed, this result motivated the heuristic approach adopted in [13] to search for violated (S,T)-inequalities. On the other hand, in Subsection 5.2, we show that given a fractional solution x and given two disjoint (S,T)-inequalities with $|S| = 1$ we can decide in polynomial time whether there exists a violated inequality of the form (25). In other words, given two disjoint (S,T)-inequalities with $|S| = 1$, the patching procedure described in Section 3 runs in polynomial time.

5.1. Separating (S, T) Inequalities

The separation problem for (S,T)-inequalities with $|S| = 1$ can be formulated as a decision problem in the following way:

Given: A complete graph $G = (V,E)$, a vertex $s \in V$, and a (fractional) solution $x \in \mathbb{R}^{|E|}$.
Question: Is there a subset $T \subseteq V\backslash\{s\}$ such that $x(E(\{s\},T)) - x(E(T)) > 1$?

We refer to this decision problem as problem SEPST.

Theorem 5.1. SEPST is NP-complete.

Proof. We prove that SEPST is NP-complete by showing that MAXCUT (which is known to be NP-complete; see [11]) reduces to it. MAXCUT is defined as follows:

MAXCUT:
Given: A graph $H = (U,F)$, edge weights $w_{ij} \in \mathbb{N}$ for all $\{i,j\} \in F$, a positive integer K.
Question: Is there a partition of U into two disjoint sets U_1 and U_2, such that

$$\sum_{\{i,j\} \in E(U_1,U_2)} w_{ij} > K? \tag{26}$$

Given an instance of MAXCUT, construct a graph $G = (V,E)$ and a point $x \in \mathbb{R}^{|E|}$ as follows: $V = U \cup \{s\}, E = \{\{i,j\} | i,j \in V, i \neq j\}$, and $x \in \mathbb{R}^{|E|}$ is defined by

$$x_{ij} = \begin{cases} \frac{1}{K} \cdot \sum_{\{j,k\} \in F} w_{jk} & \text{if } i = s, j \in U, \\ \frac{2}{K} \cdot w_{ij} & \text{if } \{i,j\} \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Let $T \subseteq U$. Then, the following holds:

$$\begin{aligned} &\sum_{i \in T} x_{si} - \sum_{\{i,j\} \in E(T)} x_{ij} \\ &= \frac{1}{K} \cdot \sum_{i \in T} \sum_{\substack{j \in U \\ \{i,j\} \in F}} w_{ij} - \frac{2}{K} \cdot \sum_{\{i,j\} \in E(T)} w_{ij} \\ &= \frac{1}{K} \cdot \sum_{i \in T} \left(\sum_{\substack{j \in T \\ \{i,j\} \in F}} w_{ij} + \sum_{\substack{j \in U \backslash T \\ \{i,j\} \in F}} w_{ij} \right) - \frac{2}{K} \\ &\cdot \sum_{\{i,j\} \in E(T)} w_{ij} \\ &= \frac{1}{K} \cdot \sum_{i \in T} \sum_{\substack{j \in T \\ \{i,j\} \in F}} w_{ij} + \frac{1}{K} \cdot \sum_{i \in T} \sum_{\substack{j \in U \backslash T \\ \{i,j\} \in F}} w_{ij} - \frac{2}{K} \\ &\cdot \sum_{\{i,j\} \in E(T)} w_{ij} \\ &= \frac{2}{K} \cdot \sum_{\{i,j\} \in E(T)} w_{ij} + \frac{1}{K} \cdot \sum_{\{i,j\} \in E(T,U \backslash T)} w_{ij} - \frac{2}{K} \\ &\cdot \sum_{\{i,j\} \in E(T)} w_{ij} \\ &= \frac{1}{K} \cdot \sum_{\{i,j\} \in E(T,U \backslash T)} w_{ij}. \end{aligned}$$

It is clear that there exists $T \subseteq U$ satisfying $x(E(\{s\},T)) - x(E(T)) > 1$ if and only if there exists a partition of U into U_1 and U_2, satisfying (26). This completes the proof of the theorem. ∎

Remark 1. Although, in general, it is not true that when it is NP-hard to separate over some class of inequalities called C it is NP-hard to separate over a class $D; djC$, the reduction in the above proof can be adjusted to show that it is NP-hard to separate over (S,T)-inequalities with fixed $|S|$ as well as over weighted (s,T)-inequalities. Notice, however, that the complexity of separating (S,T)-inequalities (thus without fixing $|S|$) remains unanswered here.

Remark 2. Notice that, in practice, the given fractional solution x in the input of SEPST will satisfy the triangle inequalities [if not, one probably would not look for violated (S,T)-inequalities, but instead add the violated triangle inequality]. Thus, it makes sense to investigate whether we can modify the reduction above so that the x-vector constructed satisfies the triangle inequalities. To see that this is indeed possible, consider MAXCUT with the following restrictions:

(1) $w_{ij} = 1$ for all $\{i,j\} \in F$ and 0 otherwise,

(2) $\sum_{\{i,j\}\in E(U_1,U\setminus U_1)} w_{ij} \le K$ for all $U_1 \subseteq U$ with $|U_1| \le 2$,
(3) all vertices $i \in U$ have degree at least 2, and
(4) $K \ge 4$.

It is well known that MAXCUT is NP-complete if $w_{ij} = 1$ for all $\{i,j\} \in F$ (see [11]). Checking if there exists a subset $U_1 \subseteq U$ with $|U_1| \le 2$ such that the cut $E(U_1, U\setminus U_1)$ has value more than K can be done in polynomial time. Hence, MAXCUT with restrictions (1) and (2) is still NP-complete. Restriction (3) is no real restriction. If there is a vertex in U with degree 1, say vertex i is only incident to edge $\{i,j\}$, and this edge does not belong to the cut, then moving vertex i to "the other side" of the cut increases the value of the cut by 1. Hence, vertex i and edge $\{i,j\}$ can be deleted from H, if at the same time K is replaced by $K-1$.

For fixed K, instances of MAXCUT with all edge weights equal to 1 can be solved in $O(|E|^K)$ time. Hence, instances of MAXCUT with restrictions (1)–(3) and $K \le 3$ can be solved in polynomial time. Since MAXCUT with restrictions (1)–(3) is NP-complete, this implies that MAXCUT with restrictions (1)–(4) is also NP-complete.

Let us now argue that, using these restrictions, the vector $x \in \mathbb{R}^{|E|}$ satisfies all triangle inequalities. Let $i,j \in U, i \ne j$. The triangle inequality $x_{si} + x_{sj} - x_{ij} \le 1$ is equivalent to

$$\frac{1}{K} \sum_{\substack{k\in U\\ \{i,k\}\in F}} w_{ik} + \frac{1}{K} \sum_{\substack{k\in U\\ \{j,k\}\in F}} w_{jk} - \frac{2}{K}\cdot w_{ij} \le 1. \tag{27}$$

The left-hand side of (27) is exactly $1/K$ times the value of the cut $E(\{i,j\}, U\setminus\{i,j\})$, and due to restriction (2), this cut has value at most K. Hence, x satisfies the triangle inequality $x_{si} + x_{sj} - x_{ij} \le 1$.

The triangle inequality $x_{si} - x_{sj} + x_{ij} \le 1$ is equivalent to

$$\frac{1}{K} \sum_{\substack{k\in U\\ \{i,k\}\in F}} w_{ik} - \frac{1}{K} \sum_{\substack{k\in U\\ \{j,k\}\in F}} w_{jk} + \frac{2}{K}\cdot w_{ij} \le 1. \tag{28}$$

By restriction (2), we have $\sum_{k\in U:\{i,k\}\in F} w_{ik} \le K$. By restriction (3), we have $\sum_{k\in U:\{j,k\}\in F} w_{jk} \ge 2$. This implies that inequality (28) is indeed satisfied.

Now, let $k \in U\setminus\{i,j\}$. One easily verifies that all triangle inequalities with respect to $\{i,j,k\}$ are satisfied using restrictions (1) and (4). Summarizing, SEPST remains NP-complete even if the vector $x \in \mathbb{R}^{|E|}$ satisfies all triangle inequalities.

5.2. *Disjoint Patching of 2-Partition Inequalities*

Here, we consider the separation of the facets induced by (25), that is, the facets obtained from the disjoint patching of two 2-partition inequalities with right-hand side equal to 1. This separation problem can be formulated as a decision problem in the following way:

Given: A complete graph $G = (V,E)$, a (fractional) point $x \in \mathbb{R}^{|E|}$, and two 2-partition inequalities with the right-hand side equal to 1, $a^T x \le 1$, and $b^T x \le 1$.
Question: Does a disjoint (S,T)-patching of these two inequalities exist, separating x and P?

We refer to this decision problem as SEPPATCH.

Theorem 5.2. SEPPATCH is solvable in polynomial time.

Proof. We prove that SEPPATCH is polynomially solvable by showing that it reduces to finding a minimal cut in a directed graph, which can be solved in polynomial time (see, e.g., [1]). Let V_a (V_b) be the vertex set of the support of $a^T x \le 1$ ($b^T x \le 1$), and let s (t) be the center of the inequality. The separation problem is equivalent to finding a partition of $V_a\setminus\{s\}$ into U_a and W_a and a partition of $V_b\setminus\{t\}$ into U_b and W_b, such that

$$(a+b)^T x + x_{st} - \sum_{j\in U_b} x_{sj} - \sum_{i\in U_a} x_{ti} - \sum_{\{i,j\}\in E(W_a,W_b)} x_{ij} > 2, \tag{29}$$

or showing that no such partitions exist.

Construct a digraph $D = (V', A)$ with $V' = V_a \cup V_b$ and

$$A = \{(s,u) | u \in V_b\setminus\{t\}\} \cup \{(u,v) | u \in V_b\setminus\{t\}, v \in V_a\setminus\{s\}\} \cup \{(v,t) | t \in V_a\setminus\{s\}\}.$$

Let the capacity of an arc $(i,j) \in A$ be equal to the value x_{ij} of the corresponding edge in G. Rewriting (29) yields

$$\sum_{j\in U_b} x_{sj} + \sum_{i\in U_a} x_{ti} + \sum_{\{i,j\}\in E(W_a,W_b)} x_{ij} < (a+b)^T x + x_{st} - 2. \tag{30}$$

The right-hand side of (30) is fixed, whereas the left-hand side is exactly the value of the $s-t$ cut $E(\{s\} \cup U_a \cup W_b, \{t\} \cup U_b \cup W_a)$ in D (see Fig. 10). If the minimum $s-t$ cut in D is less than $(a+b)^T x + x_{st} - 2$, then we have found a separating disjoint (S,T)-patching. Otherwise, no such patching exists. ∎

6. COMPUTATIONAL EXPERIMENTS

In this section, we present a simple cutting plane algorithm based on the results of the previous sections and we report on its performance on the instances described in Subsection 1.1. We are primarily interested in the question of to what extent the polyhedral results in Sections 2 and 3 help in solving these instances. To at

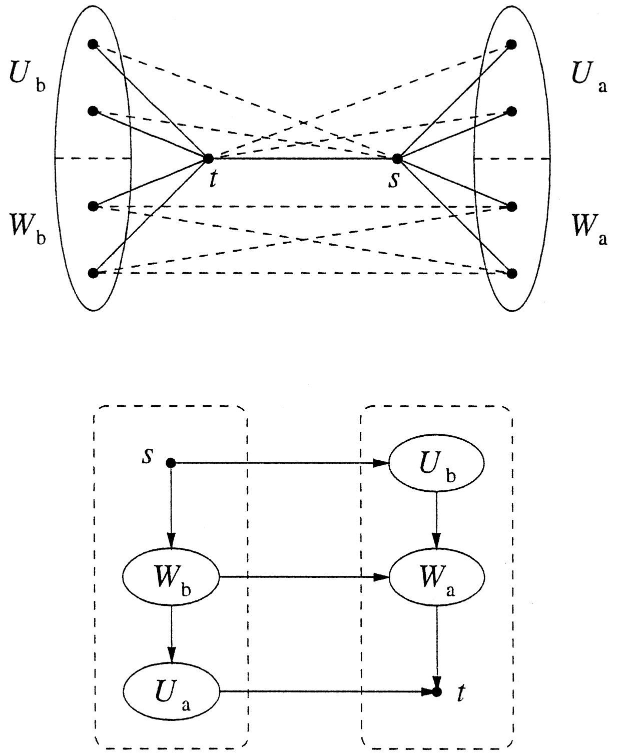

FIG. 10. The correspondence between a disjoint (S, T)-patching and a cut in a digraph.

least partially answer this question, we implemented a cutting plane algorithm that uses five of the previously described classes of facets.

For each of these classes, we implemented an exact or heuristic algorithm for finding violated inequalities. We used the following classes of facet defining inequalities:

1. Triangle inequalities; we implemented a straightforward exact (enumerative) algorithm for finding violated triangle inequalities.
2. 2-Partition inequalities; we implemented the heuristic algorithm for finding violated 2-partition inequalities with right-hand side 1, as described by Grötschel and Wakabayashi [13].
3. Weighted (s, T)-inequalities; we implemented a heuristic algorithm similar to the one used for finding violated 2-partition inequalities (see Subsection 2.1.1).
4. Stable set inequalities; we implemented an exact (enumerative) algorithm for finding cutting planes in a subclass of the stable set inequalities, namely, the wheel inequalities with right-hand side 2 (see Subsection 2.1.2).
5. Facets obtained from disjoint (S, T) patching; we implemented an exact (enumerative) algorithm for finding violated inequalities obtained from the disjoint patching of two triangle inequalities (a subclass of inequalities (25); see also Fig. 8).

The cutting plane algorithm starts with optimizing the objective function subject to the lower-bound constraints. In each iteration of the algorithm, it is checked whether the current solution is integral and feasible. If so, we are done. If not, inequalities with a large slack are deleted from the current formulation, and new violated inequalities are added to the linear program. If no violated inequalities are found the algorithm stops (we did not implement branching, since we are interested mainly in the quality of our facets in a cutting plane algorithm). Otherwise, violated inequalities are added to the current formulation, and the new linear program is solved again. These linear programs were solved with `CPLEX 2.1` on a Silicon Graphics Indigo workstation.

The inequalities are searched in the five classes mentioned above in the given order, and a class is searched only if no violated inequalities were found in one of the previous classes. We never added more than 400 cutting planes in one iteration, in order to keep the linear program within reasonable bounds. If, in three consecutive iterations, the improvement in the objective value is less than 1%, we searched for violated inequalities in each of the classes until either we found 400 cutting planes in total or all classes were searched. With respect to the topic of running times, we restrict ourselves to the following general remark: Since our main interest lies in determining the potential usefulness of the inequalities described here, no attempt was made to optimize or even record running times of the algorithm. We simply imposed an upper limit of 500 iterations for the cutting plane algorithm to keep running times manageable. The results are depicted in Table 2 (cf. with Table 1).

The first column denotes the name of the instance together with a reference, and the second column denotes the size of the instance. The column labeled z_{LP} gives the optimal LP value (which uses only the triangle inequalities). The column labeled z_{GW} gives the optimal LP value after we ran the cutting plane algorithm as described by Grötschel and Wakabayashi [13], that is, the algorithm with triangle inequalities and 2-partition inequalities as

TABLE 2. The results of the cutting plane algorithm.

Instance	Size	z_{LP}	z_{GW}	z_{CPA}	LS
KKV [18]	9×15	23.0000 (i)	—	—	23
SUL [30]	20×11	48.0000	46.5833 (71%)	46.0000 (i)	46
SEI [29]	11×22	55.6667	54.1964 (88%)	54.0000 (i)	54
LES [19]	13×25	42.0000	36.6256 (41%)	34.9571 (54%)	29
MCC [21]	24×16	56.6667	51.1740 (51%)	46.9844 (91%)	46
BOC [4]	16×43	82.0000	75.3628 (37%)	71.6947 (57%)	64
GRO [12]	23×20	75.3333	68.5182 (34%)	63.7545 (57%)	55

TABLE 3. The number of violated inequalities.

Instance	Size	Triangle	2-Partition	Weighted (s, T)	Stable Set	Disjoint (S, T)
KKV [18]	9×15	250	—	—	—	—
SUL [30]	20×11	5911	122	1	3458	1600
SEI [29]	11×22	4096	99	2	2000	0
LES [19]	13×25	3453	28	0	0	202,745
MCC [21]	24×16	4952	38	0	0	193,872
BOC [4]	16×43	6876	78	0	0	297,773
GRO [12]	23×20	7929	48	0	0	192,443

cutting planes. The next column of the table denotes the LP value found after running the cutting plane algorithm described above. Finally, in the last column, labeled *LS*, we report the value of a feasible solution found by a simple variable depth local search procedure. In this way, we have bounds on the outcome of the cutting plane algorithms. The symbol (i) indicates that the corresponding solution is integral.

The percentages given in the column z_{GW} and z_{CPA} indicate the relative amount of the integrality gap closed by the respective algorithm, that is, $(z_{LP}-LS)/(z_{LP}-z_{GW})$ and $(z_{LP}-LS)/(z_{LP}-z_{CPA})$, respectively.

For the instance KKV, the LP-relaxation is already integral. For the instances SEI and SUL, the inequalities incorporated in the cutting plane algorithm described here suffice to solve the problem to optimality. For all other instances, the algorithm described here further narrows the gap significantly. The bound resulting from the outcome of the cutting plane algorithm for the instance MCC proves the optimality of the solution found by the local search procedure. To see which inequalities contribute to the performance of the algorithm, we report in Table 3 the number of inequalities that are added for each of the five distinguished classes. It turns out that inequalities of each class have been used by the algorithm. Moreover, the class of "patched triangles" seems particularly useful for our instances. This illustrates that the new classes of facets presented here are potentially useful in a cutting plane algorithm.

7. CONCLUSIONS

In this paper, we extended the knowledge of the facial structure of clustering problems in general and the clique partitioning polytope in particular. This was done by exhibiting new classes of facet-defining inequalities and by presenting a procedure that constructs facet-defining inequalities. Further, we characterized facet-defining inequalities with right-hand side 1 or 2 and addressed some separation issues. Finally, we demonstrated that some of the inequalities described help in improving lower bounds when running a simple cutting plane algorithm on real-life instances.

Acknowledgments

The authors thank the referees for their careful reading of the manuscript.

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